The 3D Visualization of $E_8$ using an $H_4$ Folding Matrix

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This paper will present various techniques for visualizing a split real even $E_8$ representation in 2 and 3 dimensions using an $E_8$ to $H_4$ folding matrix. This matrix is shown to be useful in providing direct relationships between $E_8$ and the lower dimensional Dynkin and Coxeter-Dynkin geometries contained within it, geometries that are visualized in the form of real and virtual 3 dimensional objects. A direct linkage between $E_8$, the folding matrix, fundamental physics particles in an extended Standard Model GraviGUT, quaternions, and octonions is introduced, and its importance is investigated and described.

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I. INTRODUCTION

Fig. 1 is the Petrie projection of the largest of the exceptional simple Lie algebras, groups and lattices called $E_8$. It has 240 vertices and 6720 edges of 8 dimensional (8D) length $\sqrt{2}$. Interestingly, in addition to containing the 8D structures of $D_8$ (aka. the rectified 8-orthoplex) and $BC_8$ (aka. the 8 demicube or alternated octeract), $E_8$ has been shown to fold to the 4D Polychora of $H_4$ (aka. the 120 vertex 600-cell) and a scaled copy $H_4\Phi[10][20]$, where $\Phi = \frac{1}{2}(1 + \sqrt{5}) = 1.618...$ is the big Golden Ratio and $\varphi = \frac{1}{2}(\sqrt{5} - 1) = 1/\Phi = \Phi - 1 = 0.618...$ is the small Golden Ratio. Fig. 2 shows the folding orientation of $E_8$ and $D_8$ Dynkin diagrams above the $H_4$ and $H_3$ Coxeter-Dynkin diagrams (respectively). The 600-cell is constructed from the combination of the 96 vertices of the snub 24-cell and the 24 vertices of the 24-cell shown in Fig. 3. The 24-cell is self-dual and contained within both $F_4$ and the triality symmetry of the $D_4$ Dynkin diagram. It is interesting to note that it is constructed from the 16 vertices of the $BC_4$ tesseract (or 8-cell or 4-cube) and the 8 vertices of it’s dual, the 4-orthoplex (or 16-cell). All of these polychora can be found within $E_8$ with the excluded 8-orthoplex. The snub 24-cell is constructed from even permutations of $\{\Phi, 1, \varphi, 0\}$. Also shown in Fig. 3 is the dual of the 600-cell, namely the 120-cell with 600 vertices and a trirectified $H_4$ Coxeter-Dynkin diagram (i.e. the filled node is moved to the other end).

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FIG. 2: $E_8$ and $D_6$ Dynkin diagrams in folding orientation with their associated Coxeter-Dynkin diagrams $H_4$ and $H_3$

4D Perspective Projections

BC4 8-Cell=4-Cube= Tesseract; Orthographically projects to a 3-Cube + 16-Cell=4-Orthoplex= Dual of 4-Cube; Orthographically projects to an Octahedron; Facets contain A3 3-Simplex = D4 Self-Dual 24-Cell

H4 600-Cell= 24-Cell+Snub 24-Cell 120-Cell= Dual of 600-Cell

FIG. 3: 4D Polychora

$$H_4^\text{fold} = \begin{pmatrix} \Phi & 0 & 0 & 0 & \varphi^2 & 0 & 0 & 0 \\ 0 & \varphi & 1 & 0 & 0 & -\varphi & 1 & 0 \\ 0 & 1 & 0 & \varphi & 0 & 1 & 0 & -\varphi \\ 0 & 0 & \varphi & 1 & 0 & 0 & -\varphi & 1 \\ \varphi^2 & 0 & 0 & 0 & \Phi & 0 & 0 & 0 \\ 0 & -\varphi & 1 & 0 & 0 & \varphi & 1 & 0 \\ 0 & 1 & 0 & -\varphi & 0 & 1 & 0 & \varphi \\ 0 & 0 & -\varphi & 1 & 0 & 0 & \varphi & 1 \end{pmatrix}$$ (1)
The specific matrix for performing this folding of $E_8$ group vertices was shown several years ago to be that of (1). Notice that $H_4^{\text{fold}} = H_4^T$ such that it is symmetric with a quaternion-octonion Cayley-Dickson like structure. Only the first 4 rows are needed for folding $E_8$ to $H_4$, but the $8 \times 8$ square matrix is useful in the rotation of 8D vectors by taking its inverse.

$E_8$ also contains the 6D structures of the 6-cube or hexeract as shown in Fig. 4. It has been shown that using rows 2 through 4 of $H_4^{\text{fold}}$ projects the 6-cube down to the 3D Rhombic Triacontahedron. This particular object is interesting in that it contains the Platonic solids including the icosahedron and dodecahedron, and has been used to describe the $\Phi$ related geometry leading to quasicrystals.

II. $H_4^{\text{fold}}$ MATRIX ANALYSIS

Projection of $E_8$ to 2D (or 3D) requires 2 (or 3) basis vectors $\{X, Y, Z\}$. We start with those in (2), which are simply the two 2D Petrie projection basis vectors of the 600-cell (aka. the Van Oss projection) as shown in Fig. 5 a), with a 3rd z basis vector added for the 3D projection. Notice the 8D basis vectors with zero in the first 4 columns (or dimensions).

\[
X = \{0.0522642, 1/4, 0, -0.404508, -0.221395, 1/4, 0, 0.404508\}
\]
\[
Y = \{0, -0.27216, -0.160853, 0.203368, 1/4, 0.27216, 0.497261, 0.203368\}
\]
\[
Z = \{-0.154508, 0.0845653, 0, -0.13683, 0.654508, 0.0845653, 0, 0.13683\}
\]
FIG. 6: $E_8$ projection showing $H_4$ and $H_4\Phi$ orthonormal face orientation in 2D and 3D perspective. Only 1220 of 6720 edges are shown in order to prevent occlusion of vertices in 3D.

Eigenvalues: $H_4^{\text{fold}}$ eVal = 2[$ST, \varphi ST$]
$H_4^{-1\text{fold}}$ eValInv = [$ST, \Phi ST$]/2

Eigenvalues of $H_4^{\text{fold}}$ eVal =

\[
\begin{pmatrix}
0 & -1 & 1 & 0 & 0 & -1 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & -1 & 0 & 0 & -1 & 1 \\
0 & 0 & -1 & -1 & 0 & 0 & 1 & 1 \\
0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0 & 1 & 0 & 0 & 0
\end{pmatrix}
\]

In addition, the $E_8$ projection basis (3) is obtained by $\{X, Y, Z\} = H_4^{-1\text{fold}}\{x, y, z\}$. On one face (or 2 of 6 cubic faces, which are the same), they project $E_8$ to its 2D Petrie projection shown in Fig. 1. On another face of this particular 3D projection is what would be found on all 6 faces of an orthonormal projection to 3D of the $H_4$ 600-cell combined with a scaled $H_4\Phi$, shown in 2D on Fig. 5 b) and in 3D in Fig. 6. It is also interesting to note the $\{X, Y, Z\}$ quaternion-octonion $\Phi$ related scaling between dimensions $\{1, 5\}$ and $\{3, 7\}$, and the $\pm$ sign pairing of $\{2, 6\}$ and $\{4, 8\}$.

In addition, the {eigenvalue, eigenvector} systemics of $H_4^{\text{fold}}$ and $H_4^{-1\text{fold}}$ relate to the general relativistic (GR) space-time metric signature in 4 as quaternion parts of the octonion vectors in 5. The eigenvector matrix is shown in 6, where $H_4^{\text{fold}} = eVec^T \cdot \text{Diag}(eVal). (eVec^T)^{-1}$. The eigenvectors of $H_4^{-1\text{fold}}$ are the same as those in $eVec$.

This pattern of eigenvalues and eigenvectors strongly suggests that $E_8$ (and $H_4$) passes through a “geometric identity” as it folds (or unfolds), respectively. This makes establishing a unit determinant of these matrices interesting. The $\text{Det}(H_4^{\text{fold}}) = (4\varphi)^3 (\Phi^2 - \varphi^4) = 37,3499...$, such that $\text{Adj}(H_4^{\text{fold}}) = \text{Det}(H_4^{\text{fold}}) \cdot H_4^{-1\text{fold}}$. Establishing the $\text{Det}(H_4^{\text{fold}}) = 1.00$ by dividing by 37.3499 is easily done. Yet, $\text{Det}(H_4^{\text{fold}})/(4\varphi)^3 (\Phi^2 - \varphi^4) \approx 0$, suggesting the rows and columns of the matrices not independant.
FIG. 7: $E_8$ Dynkin diagram with Cartan, Schlafli, and Coxeter matrices

There are several choices for the form of $E_8$, whether it be complex or split real (even or odd). For the purposes of this work, the form selected is split real even (SRE). While the basic topology of the $E_8$ Dynkin diagram is unique, it has $8! = 40320$ permutations of node ordering. The node order used here is given in Fig. 7. The 240 specific $E_8$ group vertex values are determined from the simple roots matrix $E_8^{srm}$ shown in (7). The resulting Cartan matrix and generated algebraic roots are directly dependent on these as inputs.

$$E_8^{srm} = egin{bmatrix}
2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 2 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 2 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\
0 & 0 & 0 & -1 & 0 & -1 & 2 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2
\end{bmatrix}$$

(7)

The Dynkin diagram was constructed as user input with the Mathematica “VisibLie” notebook. Fig. 7 was generated and exported from the referenced tool, as are all of the figures in this paper. It has the same node ordering as the $E_8$ Dynkin used in Fig. 2, but is now shown with the assigned physics particles with SRE $E_8$\# \{206, 194, 184, 176, 1, 169, 170, 166\} that make up the simple roots matrix row entries of (7). The Cartan matrix can be generated directly by the structure of the Dynkin diagram or from its relationship to the simple roots matrix (8). The positive $E_8$ algebra roots are generated by the Mathematica “SuperLie” package and listed with its Hasse diagram in Appendix A. The 120 positive and 120 negative algebra roots are then used to generate the SRE $E_8$ vertices using (9).

$$E_8^{Cartan} = E_8^{srm}.E_8^{srm}$$

(8)

$$E_8^{SREvertex} = E_8^{srm}.E_8^{root}$$

(9)

The $E_8$ GraviGUT Extended Standard Model Construction

Lisi has proposed an extended Standard Model (SM) GraviGUT based on an $E_8$ Lie Algebra with a fundamental physics particle associated with each of its 240 roots[11]. While the particle assignments
were modified from his original model to his current model[12], the model used here is closer to the original. It is modified slightly in order to create a complete 8-bit quantum pattern consistent with Figures 9 and 10. The complete $E_8$ vertex to particle and octonion assignments are listed for reference in Appendix B. The construction of this model is based on the $256 = 2^8$ binary pattern from the 9th row of the Pascal triangle \{1, 8, 28, 56, 70, 56, 28, 8, 1\} and its associated $Cl_8$ Clifford Algebra, shown in Fig. 8.

**FIG. 8:** SRE $E_8$ construction from Pascal Triangle, $Cl_8$ Clifford Algebra and binary permutations

```
<table>
<thead>
<tr>
<th>Anti (pP)</th>
<th>E_8</th>
<th>pType (0 1)</th>
<th>2p</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gen</td>
<td>0</td>
<td>Generations</td>
<td>3g</td>
</tr>
<tr>
<td>Spin</td>
<td>4</td>
<td>Spin 1 [1,0]</td>
<td>3g</td>
</tr>
<tr>
<td>Color</td>
<td>0</td>
<td>Color (w)</td>
<td>3g</td>
</tr>
<tr>
<td>Row</td>
<td>5</td>
<td>4</td>
<td>3g</td>
</tr>
<tr>
<td>Count</td>
<td>4</td>
<td>3g x 3g</td>
<td>2</td>
</tr>
</tbody>
</table>

$2^8 = 256 = 128 + 112 + 8 + 8$

<table>
<thead>
<tr>
<th>E_8</th>
<th>Excluded Dim-AntiDim</th>
</tr>
</thead>
<tbody>
<tr>
<td>Binary: 1:1 w/</td>
<td></td>
</tr>
<tr>
<td>Permutations [{1,1,0,0,0,0,0,0}]</td>
<td></td>
</tr>
<tr>
<td>Permutations [{0,0,0,0,0,0,1,1}]</td>
<td></td>
</tr>
<tr>
<td>Permutations [{1,0,0,0,0,0,1,1}]</td>
<td></td>
</tr>
</tbody>
</table>
```

**FIG. 9:** Particle flavor counts given quantum number assignments

```
<table>
<thead>
<tr>
<th>Anti (pP)</th>
<th>F_4</th>
</tr>
</thead>
<tbody>
<tr>
<td>pType = 1</td>
<td>q4</td>
</tr>
<tr>
<td>pType = 0</td>
<td>q4</td>
</tr>
<tr>
<td>q4 = [D4 \times (\pm u_\phi, \pm u_\delta, \pm u_\epsilon)]</td>
<td></td>
</tr>
<tr>
<td>Orthos =</td>
<td></td>
</tr>
<tr>
<td>Gen</td>
<td>0</td>
</tr>
<tr>
<td>Spin</td>
<td>4</td>
</tr>
<tr>
<td>Color</td>
<td>0</td>
</tr>
<tr>
<td>Row</td>
<td>5</td>
</tr>
<tr>
<td>Count</td>
<td>4</td>
</tr>
</tbody>
</table>
```

**FIG. 10:** Particle flavors in row / column groups with boson {group} coloring based on Lie group assignments ($F_4$, $F_4^4$, $D_4$ & $G_2$, $G_2^4$)

In this model, the 16 particles associated with columns 2 and 8 of the 9th row of the Pascal triangle \{8, 8\} are excluded as dimensional generators from the permutations of \{\pm 1, 0, 0, 0, 0, 0\}. These excluded particles are associated with the 8-orthoplex (dual of the 8-cube with 256 vertices). While the positive generators are added to the “dimension count” of $E_8$, they are not included as vertices per se, but they do show up in the projections as the axis of the basis vectors. This leaves $E_8$ with its 120 positive roots and 120 negative roots in the other 7 columns of the Pascal triangle.

The SRE $E_8$ roots are defined by combining the 112=\{56, 56\} integer roots of Lie group $D_8=SO(16)$ with 128=\{1, 28, 70, 28, 1\} half integer roots of Lie group $BC_8=Sp(16)$. Specifically, $D_8$ contains all permutations of \{\pm 1, \pm 1, 0, 0, 0, 0, 0\} and $BC_8$ contains all permutations of \{\pm 1, \pm 1, \pm 1, \pm 1, \pm 1, \pm 1, \pm 1\}/2 with an even number of plus signs (an 8 demicube or even 7-cube).
There are 48 assigned $D_8$ integer bosons and only 128 $C_8$ half-integer vertices available. Yet, with $192 = 64 \times 3$ generation fermions in SM, the meaning or validity of assigning a generation of fermions to the remaining 64 $D_8$ integer vertices has been hotly debated[7]. In this model, the remaining 64 integer vertices are assigned to the 2nd generation fermions. For a complete reference of particle assignments, see Appendix B.

The specific particle assignments are determined by the configuration of the particle {spin, color, generation, flavor and type} and the patterns within $E_8$. The particle type \{e, $\nu_e$\} or \{u, d\} and spin \{\bar{\nu}, \bar{\nu}, \bar{\nu}, \bar{\nu}\} are assigned or encoded in the positions or dimensions \{1, 2, 3, 4\} of each $E_8$ vertex. The generations are encoded in position \{5\}, and color in positions \{6, 7, 8\}. The antiparticle operation is simply the negation of the $E_8$ vertex (or in the binary representation “inverted” $\neg \Leftrightarrow 1$ as shown in Fig. 8). It should be noted that although the positive roots of the algebra are not all assigned as “particles”, the negation of the root does represent the “anti” particle operation on the assigned particle. The charge calculation for the particles is obtained by $Q = E_{8\text{SRE vertex}} \{0, 0, 0, -3, 0, 1, 1\} / 3$. This provides accurate results for the generation 0 bosons and 1st and 3rd generation fermions. It shows interesting deviations for some of the 2nd generation fermions that have been assigned to the $D_8$ integer vertices.

It is also helpful to note that the entire binary and SRE vertex list (as constructed in Fig. 8 and listed in Appendix B) is lexicographically ordered from negative to positive with a left-right and bottom-top mirroring about the middle, between the 128th and 129th of 256 vertices, which are the $\bar{\nu}$ tau neutrinos $\nu_e$ and $\bar{\nu}_\tau$. Also of interest are the first and last vertex particles which are rows \{1, 9\} of the Pascal Triangle with all 0 or all 1/2 entries. These are the $\bar{\nu}$ electron neutrinos $\nu_e$ and $\bar{\nu}_\tau$. This integrated model aligns well with the idea that it is associated with (T)ime reversal in the Charge-Parity-Time (CPT) conservation laws and points to the special consideration needed for the right handed neutrinos in the SM.

III. THE QUANTUM BIT-WISE PARTICLE ASSIGNMENTS

The 1:1 bit-wise correspondence of a particle’s quantum number assignments is a big-endian (left most significant) zero-based 8 dimensional vector \{0-7\}. The assignments are \{1 anti-particle bit=\{a\} (p/$\bar{p}$), 1 particle bit=\{p\} (e/$\nu$ leptons or u/d quark), 2 color bits=\{c1, c0\} (w=0 or none/r/g/b), 2 spin bits=\{s1, s0\} (\bar{l}, \bar{\nu}, \bar{l}, \bar{\nu}), and 2 generation bits=\{g1, g0\} (0=bosons/e/$\mu$/\tau)\} or simply \{a, p, s1, s0, c1, c0, g1, g0\}. The bold type face indicates quantum assignments which are not only allocated to an SRE $E_8$ vertex dimension as described above, but are exhibited in the inherent structural symmetry of the $E_8$ algebra, group or lattice. These bold bits are used to define the 3 bit structure of (10) associated with the $E_8$ Dynkin diagram in Fig. 7.

$$E_8 \text{ Dynkin}_{4 \text{ bit}} = \begin{pmatrix} 3\text{bit} = \begin{array}{c} \text{Boson} \quad \text{Fermion} \\ 0 \quad 1 \end{array} \\ 4 = 2^2 \quad \begin{array}{cc} \text{Gen 0-2} & \text{Gen 1-3} \\ p & e/d \\ 1 = 2^0 \quad a & p & \bar{p} \end{array} \end{pmatrix} \quad (10)$$

$$\text{Physics}_{\text{rot}} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{2} & -\sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{2} & \sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \sqrt{\frac{2}{3}} \end{pmatrix} \quad (11)$$
FIG. 11: $E_8$ showing the Petrie projection face orientation in 2D and 3D perspective with vertices as physics particle assignments. Vertex shape, size, color/shade are assigned based on extended Standard Model particle assignments. Only 1220 of 6720 edges are shown in order to prevent occlusion of vertices in 3D perspective.

Fermionic Triality

$$\text{Triality}_{\text{rot}} = \frac{1}{2} \begin{pmatrix}
-1 & 1 & 1 & -1 & 0 & 0 & 0 & 0 \\
-1 & 1 & -1 & 1 & 0 & 0 & 0 & 0 \\
-1 & -1 & 1 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & -1 & -1 & -1 \\
0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 \\
0 & 0 & 0 & 0 & 1 & -1 & 1 & -1 \\
0 & 0 & 0 & 0 & 1 & -1 & -1 & 1
\end{pmatrix}$$

(12)

Bosonic Triality

$$\text{Triality}_{\text{rot}} = \frac{1}{3} \begin{pmatrix}
3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -3 & 0 & 0 & 0 & 0 \\
0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -3 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -2 & -1 & 2 \\
0 & 0 & 0 & 0 & 0 & 2 & -1 & -1 \\
0 & 0 & 0 & 0 & 0 & -1 & 2 & -1
\end{pmatrix}$$

(13)

As already described for the $E_8$ vertices, the $\{a\}$ bit splits the 128 particles from 128 anti-particles. The $\{g0\}$ bit splits the generation 0 boson family of 128 (=112 integer roots of $D_8+16$ excluded integer roots of the 8-orthoplex) from the 128 half integer root (and half integer spin) of $C_8$ fermions. The $\{p\}$ bit splits all particle families into two types, referenced in the leptons as electron and neutrino types, while the quarks are designated by up and down types. It splits the integer bosons into 2 types as well, which is a key feature of this model over the original Lisi model.

These differences are most easily seen in the $8 \times 8$ rotation matrix used for transforming SRE coordinates to physics coordinates (11) and those matrices used in identifying fermionic (12) and bosonic (13) triality transformations. Just as the $E_8$ to $H_4$ folding matrix has symmetric quaternion quadrants in the octonion matrix, the physics and triality rotation matrices are divided by an upper left quadrant affecting the SRE $\{1-4\}$ spin positions and a lower right quadrant affecting the SRE $\{5-8\}$ generation-color positions. As a matter of fact, the physics rotation clearly operates on the SRE $E_8$ vertices by pairing them into 4 sets, specifically $p_{\text{type}} \{1,2\}$, spin $\{3,4\}$, generation $\{5,6\}$ and color $\{7,8\}$. This physics rotation is more dramatically shown in the next section on triality.

Visualizing this $E_8$ based physics model by projection to 2D and 3D with vertex shape, size and color assigned based on the described patterns is now possible. Using the direct relationship to lower dimensional geometry symmetries provided by the folding matrix provides the flexibility to select the
The Lisi model also demonstrates a consistency with the bosons and fermions that is related to the triality relationships within $E_8$. This is shown in Fig. 12 with blue triality lines linking the 3 generations of each fermion using (12). Applying the triality rotation matrix as a dot product against an SRE vector gives the 2nd generation fermion particle. Applying it again gives the 3rd generation. Applying it a 3rd time returns to the 1st generation fermion. The bosons are also involved in triality relationships as well using (13), rotating through red, green, and blue particle color assignments.

It is interesting to note that the quarks $\{r/g/b, p/\bar{p}\}$ are all located on 6 corresponding dual concentric circles around the center. The leptons are hexagonal “Star of David” patterns in the center, while the bosons are in single or dual hexagonal rings radiating from the center.
The axis shown in Fig. 12 are rotated to physics coordinates using (11), which puts the basis vectors (14) on the projected (H)orizontal and (V)ertical axis. It seems to clarify dimensional identities as well. For example, when the \( \{1, 2, 3\} \) dimensions are moved (i.e. using axis locators in the tool), all vertices change positions except the \( p_{\text{type}} = 0 \) bosons \( \{g \text{ gluons, } x_n \Phi\} \). Moving dimension \( \{4\} \) preserves these as well as the \( \hat{l} \) and \( \hat{r} \) quark positions. Moving the dimensions \( \{5, 6\} \) preserves these, except now the row 4 \( p_{\text{type}} = 0 \) bosons \( \{x_n \Phi\} \) emerge from the 6 triple overlap points at center of the quark’s concentric rings (the intersection of the gluons triality lines). And finally, the \( \{7, 8\} \) dimensions in physics can be identified with quark color, as \( \{7\} \) preserves the blue quark positions, while \( \{8\} \) moves the dual concentric rings of quarks while preserving their relative positions within the rings. It is interesting to note that the dimensions \( \{6, 7, 8\} \) are appropriately labeled \( \{r, g, b\} \) in SRE coordinates, since in this projection the SRE math coordinates are located at the aforementioned 6 triple overlap points at center of the quark’s \( \{\bar{r}, g, \bar{g}, b, \bar{b}\} \) concentric rings (the intersection of the gluons triality lines).

\[
H = \begin{bmatrix}
2 - \frac{4}{\sqrt{3}} & 0 & 0 & \sqrt{2} - \sqrt{\frac{2}{3}} & 0 & 0 & \sqrt{2} & 0 \\
0 & 0 & \sqrt{2} - \sqrt{\frac{2}{3}} & 0 & 0 & 0 & -\sqrt{2}
\end{bmatrix}
\]

IV. \( E_8 \) TO \( H_4 \) FOLDING’S APPLICATION TO THEORETICAL PHYSICS

\( H_4^{\text{fold}} \) provides a new and more direct relationship between \( E_8 \) and its lower dimensional geometric objects such as \( H_4 \). This has allowed for improvements to \( E_8 \) related physics models, such as those of Lisi[11]. This theoretical model is shown to provide \( E_8\text{_{perm}} \) assigned particles as fundamental building blocks for generating the rest of the 240 \( E_8 \) vertex mapped particles\[14\].

Those specific particle assignments now also include their association to 8 bitwise quantum numbers, which are: the anti-particle bit, the particle type bit \( \{e, \nu_e\} \) or \( \{u, d\} \), 2 spin bits \( \{\hat{l}, \hat{r}, \hat{\bar{l}}, \hat{\bar{r}}\} \), 2 color bits \( \{w = 0, r, g, b\} \), and 2 generation bits \( \{0, 1, 2, 3\} \). This capability of mapping specific particles to \( E_8 \) has allowed the verification of related results\[9\].

This has also allowed for improved charge calculations in Lisi’s extended GraviGUT integrated Standard Model (SM)\[12\]. This was done through the analysis of variations associated with a particular association of generation 0 bosons and generation 1-3 fermions with \( H_4 \) and \( H_4\Phi[19] \).

\( E_8 \) has also been shown to be related to an 8 dimensional Charge-Parity-Time (CPT) construct for a Theory of Everything (ToE)\[13\]. This now includes particle mass predictions such as a Higgs mass of 124.443...\( \text{GeV}[15] \), which is within the current error bars of the LHC CMS experiment results for a discovered Higgs particle mass of 124.70 \( \pm 0.31 \) (stat) \( \pm 0.15 \) (syst) \( \text{GeV}[6] \). More particle mass predictions based on the model have been found to be within standard experimental error. Other mass predictions are suggested by the features of this integrated geometry based physics model and are an active part of the author’s research on the topic.

V. OCTONIONS AND \( E_8 \) WITH \( H_4 \) FOLDING

In addition to mapping extended SM particle quantum bits between \( H_4 \) and the SRE \( E_8 \) vertices and algebra roots, they have been mapped\[17\] to the 480 unique permutations of octonions\[5\] and the 3840 split octonions\[18\] through a common pattern associated with the quantum bits. This octonionic mapping provides valuable insight into related theoretical physics models.

Fano plane triads
\[
\{\{1, 2, 4\}, \{1, 3, 7\}, \{1, 5, 6\}, \{2, 3, 5\}, \{2, 6, 7\}, \{3, 4, 6\}, \{4, 5, 7\}\}
\]

Flattened triads
\[
\{1, 4, 2, 3, 7, 5, 6\}
\]

Mask bits
\[
\{0, 0, 0, 0, 0, 0, 0\}
\]

The double cover of the 240 vertices requires the addition of a 9th “flip” bit that operates on the
Fano plane representation by reversing two of the midpoint nodes in the Fano plane triads (i.e. those with circular directed edges in Fig. 13). These node numbers are always indicated by the 2nd and 3rd columns of the flattened triads as shown in (15). They are bolded when reversed (or “flipped”), which shows this particular octonion has the flip bit set. The flattened triad is simply created by taking in sequence the numbers from the first triad along with the last two numbers in the 2nd and 3rd triads. It operates to define the node numbers for each canonical position of the Fano plane mnemonic.

As it was for the permutation of node numbers in Dynkin diagrams, there are many permutations of node number and arrow direction in the octonion Fano plane which are equivalent. What is important is the representation of the triads given in (15). This particular set of triads is equivalent to that used in Baez’ work on octonions[4].

![Fano plane representation](image-url)
which are the positive (and negative) generators commonly associated with the 8-orthoplex with 16
In order to make a valid octonion, each fpi gets one of 8 possible 7-bit sign masks (sm) applied
30 = 480
8
12
15
2
(18)
(17)
sm = \{00, 07, 19, 1E, 2A, 2D, 33, 34\}
sm2fpi = \{5, 8, 4, 3, 7, 6, 3, 2, 6, 5, 1, 4, 6, 7, 3, 8, 6, 3, 1, 6, 6, 2, 3, 5, 8, 4, 3, 7, 6\}
(18)
There are 30 canonical sets of 7 triads indexed with a Fano plane index (fpi) in (16). As in \(E_8\) with
16 of the \(2^8 = 256\) binary representations excluded from the group, there are 32 excluded octonions from the \(2^9 = 512\). As in \(E_8\), excluded particles are associated with the color=0, generation=0 (bosons) which are the positive (and negative) generators commonly associated with the 8-orthoplex with 16 permutations of \(\{\pm 1, 0, 0, 0, 0, 0, 0\}\).
In order to make a valid octonion, each fpi gets one of 8 possible 7-bit sign masks (sm) applied (17). Since each sm can be “inverted” (0 ↔ 1 as we do with the anti-particle quantum bit), this gives 16 * 30 = 480 octonion permutations.
The sign mask operates on the triads by reversing the 2nd and 3rd numbers from canonical (numerical) order when the mask bit is set on that triad’s position. Each sign mask operation acting on the 30 fpi’s can be permuted in consistent ways to produce the many isomorphic sets of 480 octonions. Since they are bit-wise operations, the sign masks use hexadecimal notation with the first bit always 0. It is interesting to note that there are only 2 octonions that use a sign mask of 00H. The one shown and another discovered by Dixon[8].
The 8 sets of 8 sm are assigned to the 30 fpi given in (18), so if fpi=1, the 5th sm group is selected. Since the octonion in Fig. 13 has fpi=11 and the 11th sm2fpi=1, this means sm bits will index to the 1st sm group in (17).

Assigned Particle and Quantum bits
SRE $E_8\#177 = s^{1/3} r^L$
Quantum bits \( \{apcsssgg\} \)
\( \{00010010\} \) (19)

In this integrated system of $E_8$, particles, and octonions, the 4 bits that make up the 16 possible sign masks are associated with the 4 quantum particle bits \( \{\text{anti, ptype}, \text{and 2 spin bits}\} \). Looking deeper at the space-(P)arity orientation pattern, where pitch and roll rotations are associated with the up/down and left/right spin bits, a conjecture is made that the ptype bit can be thought of as a 3rd spin bit, giving a 3rd spin type which we might call “in/out” or yaw rotation. Since according to (19) the octonion in Fig. 13 has ptype=0, and spin=00 or \( L \), this means sm=1, so it gets the 1st sign mask of the 1st sm group, giving sm=00H.

The assignment of the 30 fpi’s is based on the 4 color and generation bits that are not both 0 (or excluded) giving $2^4 - 1 = 15$. It is not simply a naive index created by simply adding the flip bit to the index. The extra logic needed to index $2^4 \cdot 15$ is based on a pattern discovered that relates how the anti and flip bits operate across the generation and color bits. Specifically, this pattern differentiates the 128 1/2 integer $E_8$ vertices associated with the $BC_8$ group (2 generations of fermions) from the integer 112 vertices in the $D_8$ group (bosons plus an anomalous generation=2 set of fermions).

VI. CONCLUSION

![Figure 14: The $E_8$ to $H_4$ 3D projection model used to laser etch optical crystal](image)

In terms of mathematical symmetry representing the beauty of Nature, $E_8$ is one of the most beautiful. It contains a wealth of symmetries, including those of 2D projections, 3D polyhedrons, 4D polychora, and those up to 8D. An SRE $E_8$ to $H_4$ folding matrix was determined and used to fold $E_8$ to the 120 4D vertices of the $H_4$ 600-cell and 120 vertices of $H_4\Phi$. A direct relationship between the simple roots matrix and theoretical physics models was introduced. In addition to the mapping for particles, a direct relationship to the 480 unique octonion permutations was also shown.

The traditional 2D Petrie projections of high dimensional geometry were extended by adding a carefully chosen third basis vector and generating 3D objects in either orthogonal or perspective views. The folding matrix was shown to generate these basis vectors used in projecting the $E_8$ vertices. These projected 3D objects can be realized as 3D models, which allow for their realization as animated rotations, models laser etched in optical crystal, and in some cases 3D printed in plastic or even metal as in Fig. 14.

In addition, these new mathematical relationships and visual representations have been used to verify and improve grand unified theories which rely on these structures.

Acknowledgments

I would like to thank my wife for her love and patience and those in academia who have taken the time to review this work.
Bibliography

## VII. APPENDIX A

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### VIII. APPENDIX B

<p>| Symbo | 2D/3D Shape | Groups | Particle | Quantum Bits | Energy / S/ S/ | Coordinates | Alpha Root | Weight (Opp) | Dirac Spinor Elements Number | B | 4s | 4p | 4d | 4f | 5s | 5p | 5d | 5f | 6s | 6p | 6d | 6f | 7s | 7p | 7d | 7f | 8s | 8p | 8d | 8f | 9s | 9p | 9d | 9f | 10s | 10p | 10d | 10f | 11s | 11p | 11d | 11f | 12s | 12p | 12d | 12f | 13s | 13p | 13d | 13f | 14s | 14p | 14d | 14f | 15s | 15p | 15d | 15f | 16s | 16p | 16d | 16f |</p>
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