

In[3465]:= (*** Furey C \otimes O ***)

This is an analysis of several papers from Cohl Furey related to using octonions in the assignment of Boson and Fermion particles to Clifford algebras and group symmetries.

This .pdf is a direct output from my Mathematica Notebook. This notebook has code built in to operate symbolically on Lie algebras/groups derived from E8 (240 vertices + 8 generators and 8 "anti-generators"), organized by the 256 entries in the 9th row of the Pascal Triangle. It also allows for the visualization of Hasse diagrams from Cartan matrices created by entering or selecting the corresponding Dynkin (and/or Coxeter-Dynkin) diagrams. This also integrates operations using any of the 480 possible octonion multiplication tables, as well as visualizing their Fano plane (or Fano cube) mnemonic based on the 7 triads selected. These 480 octonion tables are algorithmically assigned to the 240*2 E8 vertices (where the 2x factor uses a "flip" state (or bit) to indicate that the Fano plane and its' corresponding bit masks and triads are changed). In addition to the multiplication tables, it uses Cayley-Dickson procedures to generate the Sedenions. It uses the proper built-in Mathematica complex and quaternion math functions, as well as defines conjugation, commutation, Der(ivation) matrix/vector operations to get to the 14 null G2 automorphisms (and the 21-14=7 non-Nullspace elements). My notebook also has defined a function that tests the non-associativity of octonion math as well as testing the zero divisors of sedenion math.

So to set up the analysis, we select the Fano plane used in these papers, which happens to be the same one used by Baez in his work, which is in position 224 "flip→True" in my assignment of the 480 combinations. The algorithm to visualize the Fano plane constructs it differently than the paper, but the triads are correct.

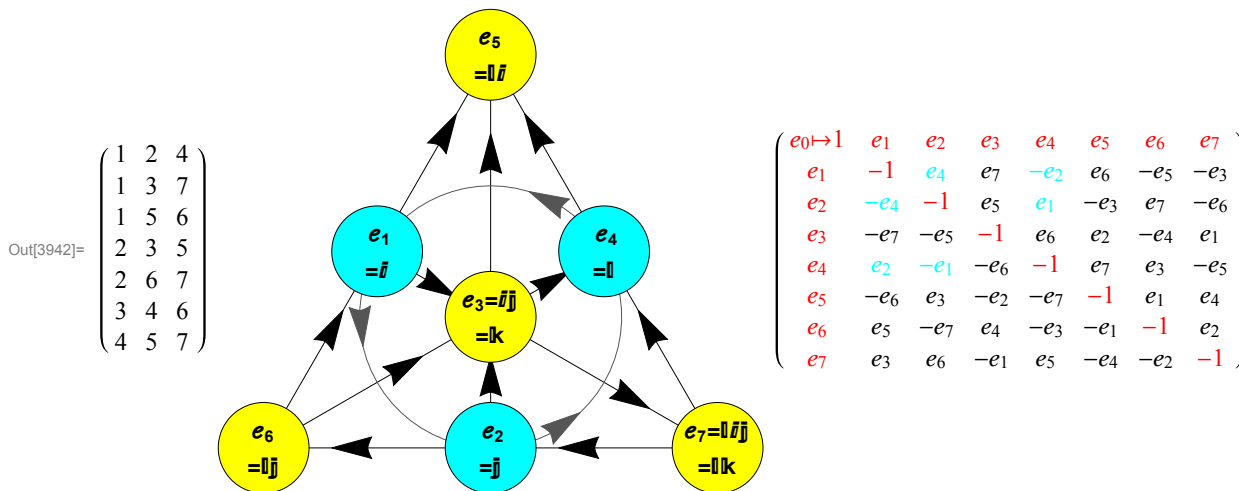
In[3466]:= (*** Cohl Furey's Octonion = Baez's octonion ***)

```
In[3467]:= Position[(setFM[\#, True];  
             StandardForm[triads]) & /@ Range@256,  
             {{1, 2, 4}, {1, 3, 7}, {1, 5, 6}, {2, 3, 5}, {2, 6, 7}, {3, 4, 6}, {4, 5, 7}}]
```

Out[3467]= (224 1)

In[3468]:= **setFM**[**224, flip = True**];

```
In[3942]= Row@{triads, fanoPlane[[1, 1, 2]], fmDispe}
```



```
In[3667]= Clear@SmallCircle
```

```
In[3662]= (**)
SmallCircle[a_, 1] := a;
SmallCircle[1, b_] := b;
```

```
In[3668]= (**)
SmallCircle[a_List, b_List] := octonion@octProduct[a, b];
SmallCircle[a_, b_Quaternion] := a ° quat2oct@b;
SmallCircle[a_Quaternion, b_] := quat2oct@a ° b;
SmallCircle[a_, b_Complex] := a ° ToQuaternion@b;
SmallCircle[a_Complex, b_] := ToQuaternion@a ° b;
SmallCircle[a_, b_] := oct2List@a ° oct2List@b;
```

```
In[3479]= oct
```

```
Out[3479]= {e1, e2, e3, e4, e5, e6, e7, e0}
```

```
In[3480]= (* Octonion Chains *)
```

```
In[3481]= e1 ° e2
```

```
Out[3481]= e4
```

```
In[3482]= e2 ° e3
```

```
Out[3482]= e5
```

```
In[3483]= e1 ° (e2 ° e3)
```

```
Out[3483]= e6
```

```
In[3484]= e3 ° (e2 ° e1)
```

```
Out[3484]= -e6
```

In[3485]= $e_6 + i \circ e_2$

Out[3485]= $e_4 + e_6$

I constructed a function f to build the left (L) parenthetical multiplication precedence, which can be reversed by simply providing a "right→True" parameter to perform the right (R) operations. In addition, it takes a "right→both" parameter to both reverse the order and the parenthetical precedence. Note the use of the SmallCircle "°" for doing proper octonion multiplication. While the paper does not show the evaluation of the multiplications, I do that in order to verify the logic.

In[3486]= **Clear@f**

```
In[3487]= (* Define the parenthetical reverse nesting
with Octonion multiplication table operation ° *)
f[in_List, right_ : False] := Module[{out = 1},
Do[out = If[right === both, in[[1]] ° out,
If[right, out ° in[[1]],
Reverse[in] [[1]] ° out]], {1, Length@in}];
out];
```

Using an example from the "Octonionic Chains" paragraph of "Generations: Three Prints, in Colour", we evaluate this function first w/o a defined multiplication table to see the proper parenthetical precedence is obtained for left and right operations:

In[3664]= **f@{e₃, e₄, e₆ + i ° e₂}**

Out[3664]= $e_3 \circ (e_4 \circ (e_6 + i \circ e_2))$

In[3665]= **f[{e₃, e₄, e₆ + i ° e₂}, True]**

Out[3665]= $(e_3 \circ e_4) \circ (e_6 + i \circ e_2)$

In[3666]= **f[{e₃, e₄, e₆ + i ° e₂}, both]**

Out[3666]= $(e_6 + i \circ e_2) \circ (e_4 \circ e_3)$

While the premise that non-associativity in octonions will lead to new maps in chains is true, evaluating this example as written with the Baez octonion math table seems to lead to the same mapping. That is we get the given answer, but for both left and right precedence.

In[3491]= $e_3 \circ e_4$

Out[3491]= e_6

In[3492]= $-1 + i \circ e_7$

Out[3492]= $-e_3 - 1$

In[3493]= **f@{e₃, e₄, e₆ + i ° e₂} == f[{e₃, e₄, e₆ + i ° e₂}, True] == -1 + i ° e₇ == -e₃ - 1**

Out[3493]= True

I believe the problem with the example is the evaluation actually used a precedence (below) to get the $-1 - i \circ e_7$ result, where the $(e_3 \circ e_4) = e_6$ is evaluated non-commutatively on the right side of $(e_6 + i \circ e_2)$, not on the left as shown in the paper:

In[3494]= $(e_6 + \mathbf{i} \circ e_2) \circ (e_3 \circ e_4) == -1 - \mathbf{i} \circ e_7$

Out[3494]= True

If the intent of the right precedence procedure is to also reverse the entries as well as the parenthetical priority, the function `f` should produce the following, noting that $(e_4 \circ e_3) = -e_6$:

In[3495]= $f[\{e_3, e_4, e_6 + \mathbf{i} \circ e_2\}, \text{both}] == (1 + \mathbf{i} \circ e_7)$

Out[3495]= True

Yet this is also not the result given. Using the associator function can show the problem with this example.

`associator[a_List,b_List,c_List]:=(a*b)*c-a*(b*c);`

It's associator is zero, meaning it can't show the non-associative property as intended.

In[3496]= `associator[e3, e4, e6 + i * e2]`

Out[3496]= 0

Changing the example to use a slightly different set of octonion dimensions ($e_6 \rightarrow e_5$), we get a non-zero associator and a different chain mapping:

In[3497]= `associator[e3, e4, e5 + i * e2]`

Out[3497]= $-2 e_1$

In[3498]= $f[\{e_3, e_4, e_5 + \mathbf{i} \circ e_2\}, \text{False}] == e_1 + \mathbf{i} \circ e_7$

Out[3498]= True

In[3499]= $f[\{e_3, e_4, e_5 + \mathbf{i} \circ e_2\}, \text{True}] == -e_1 + \mathbf{i} \circ e_7$

Out[3499]= True

In[3500]= $f[\{e_3, e_4, e_5 + \mathbf{i} \circ e_2\}, \text{both}] == -e_1 + \mathbf{i} \circ e_7$

Out[3500]= True

As we found above that when dimensions are associative (`associator=0`), both the left and right precedence of `f` are the same, but the "both" combination of precedence order and dimension element reversal result is different by a minus sign from left and right. This makes sense with 3 elements, given that the parenthetical order is not going to change the result, but the octonion element pair reversal will introduce a commutative negation. If the dimensions are non-associative, the right precedence is the same as "both", and the left precedence is different by a minus sign. This makes sense with 3 elements, due to the double negation of combining a precedence reversal and the octonion element pair reversal.

See Furey ' s latest paper

"SU(3)_C x SU(2)_L x U(1)_Y (xU(1)_X) as a symmetry of division algebraic ladder operators"
for reference :

```
In[3501]:= (* Octonion Conjugation #* *)
In[3502]:= #* & /@ oct
Out[3502]= {-e1, -e2, -e3, -e4, -e5, -e6, -e7, 1}
In[3503]:= (* 4.2 Clifford algebraic structure Mf *)
In[3504]:= (* 1 element of 0-V - e0 = 1 *)
f@{oct[[8]]}
Out[3504]= e0
In[3505]:= (* 6 elements of 1-V (without the factor of i in paper) - e1+e2+e3+e4+e5+e6 *)
Sum[f@{oct[[i]]}, {i, 1, 6}]
(* The number of unique dimensional elements after octonion multiplication *)
Length@%
Out[3505]= e1 + e2 + e3 + e4 + e5 + e6
Out[3506]= 6
In[3507]:= (* 15 elements of 2-V -
e1°e2+e1°e3+e1°e4+e1°e5+e1°e6+e2°e3+e2°e4+e2°e5+e2°e6+e3°e4+e3°e5+e3°e6+e4°e5+e4°e6+e5°e6 *)
Sum[Sum[f@oct[[{i, j}]], {j, 2, 6}], {i, 1, 6}]
Length@%
(* The total of the coefficients = the number of elements in the
original calculation minus the dimensions that cancel each other *)
Total@Abs@oct2List@%%
Out[3507]= 2 e1 + 2 e6 + 3 e7
Out[3508]= 3
Out[3509]= 7
```



```
In[3529]= (* Not the same due to the missing 4-V and 5-V for full Mf in the first section *)
%
%%%%%%%%%%
% == %%
```

Out[3529]= 20

Out[3530]= 19

Out[3531]= False

```
In[3532]= (* Adding 4-V {-2 e2+2 e3-3} and 5-V {e1+e2-e3+e4-e5-e6} from below
to first section confirms equality with this section's Mf reduced,
with adjustments removing the factor of i as in the paper *)
%%%%%%%% == Total[%%%%%%%%] + e1 + e2 - e3 + e4 - e5 - e6 - 2 e2 + 2 e3 - 3
```

Out[3532]= True

The following 0-V to 6-V reconfirms the previous 2 section's combinations (with some adjustments removing the factor of i as in the paper).

```
In[3533]= (* 0-V *)
```

```
In[3534]= f@{oct[[8]]} /. e0 -> 1
```

Out[3534]= 1

```
In[3535]= (* 1-V = 6 element *)
```

```
In[3536]= Sum[i * f@{oct[[i]]}, {i, 1, 6}]
Length@%
```

Out[3536]= $-e_2 + e_4 - e_5 + e_6 + e_7 - 1$

Out[3537]= 6

```
In[3538]= (* 2-V = 15 element -
e1 * e2 + e1 * e3 + e1 * e4 + e1 * e5 + e1 * e6 + e2 * e3 + e2 * e4 + e2 * e5 + e2 * e6 + e3 * e4 + e3 * e5 + e3 * e6 + e4 * e5 + e4 * e6 + e5 * e6 *)
```

```
In[3539]= Sum[Sum[f@oct[[{i, j}]], {j, 2, 6}], {i, 1, 6}]
Length@%
Total@Abs@oct2List@%
```

Out[3539]= $2 e_1 + 2 e_6 + 3 e_7$

Out[3540]= 3

Out[3541]= 7


```
In[3542]:= f@oct[{{#[[1]], #[[2]]}] & /@ Subsets [Range@6, {2}]
Length@%
Total@%%
Total@Abs@oct2List@%
```

```
Out[3542]= {e4, e7, -e2, e6, -e5, e5, e1, -e3, e7, e6, e2, -e4, e7, e3, e1}
```

```
Out[3543]= 15
```

```
Out[3544]= 2 e1 + 2 e6 + 3 e7
```

```
Out[3545]= 7
```

```
In[3546]:= (* 3-V = 20 element (without the factor of i as in the paper) -
```

$$e_1 \circ (e_2 \circ e_3) + e_1 \circ (e_2 \circ e_4) + e_1 \circ (e_2 \circ e_5) + e_1 \circ (e_2 \circ e_6) + e_1 \circ (e_3 \circ e_4) + e_1 \circ (e_3 \circ e_5) + \\ e_1 \circ (e_3 \circ e_6) + e_1 \circ (e_4 \circ e_5) + e_1 \circ (e_4 \circ e_6) + e_1 \circ (e_5 \circ e_6) + e_2 \circ (e_3 \circ e_4) + e_2 \circ (e_3 \circ e_5) + e_2 \circ (e_3 \circ e_6) + \\ e_2 \circ (e_4 \circ e_5) + e_2 \circ (e_4 \circ e_6) + e_2 \circ (e_5 \circ e_6) + e_3 \circ (e_4 \circ e_5) + e_3 \circ (e_4 \circ e_6) + e_3 \circ (e_5 \circ e_6) + e_4 \circ (e_5 \circ e_6)$$

$$i_1 e_1 \circ (e_2 \circ e_3) + i_1 e_1 \circ (e_2 \circ e_4) + i_1 e_1 \circ (e_2 \circ e_5) + i_1 e_1 \circ (e_2 \circ e_6) + i_1 e_1 \circ (e_3 \circ e_4) + i_1 e_1 \circ (e_3 \circ e_5) + i_1 e_1 \circ (e_3 \circ e_6) + \\ i_1 e_1 \circ (e_4 \circ e_5) + i_1 e_1 \circ (e_4 \circ e_6) + i_1 e_1 \circ (e_5 \circ e_6) + i_1 e_2 \circ (e_3 \circ e_4) + i_1 e_2 \circ (e_3 \circ e_5) + i_1 e_2 \circ (e_3 \circ e_6) + i_1 e_2 \circ (e_4 \circ e_5) + \\ i_1 e_2 \circ (e_4 \circ e_6) + i_1 e_2 \circ (e_5 \circ e_6) + i_1 e_3 \circ (e_4 \circ e_5) + i_1 e_3 \circ (e_4 \circ e_6) + i_1 e_3 \circ (e_5 \circ e_6) + i_1 e_4 \circ (e_5 \circ e_6) *$$

```
In[3547]:= Sum[Sum[Sum[i^k f@oct[{{i, j, k}}], {k, 3, 6}], {j, 2, k-1}], {i, 1, k-1}]
Length@%
```

```
Total@Abs@oct2List@%%
```

```
Out[3547]= 2 e2 - 2 e3 - 4
```

```
Out[3548]= 3
```

```
Out[3549]= 8
```

```
In[3550]:= (* i^k *) f@oct[{{#[[1]], #[[2]], #[[3]]}] & /@ Subsets [Range@6, {3}]
```

```
Length@%
Total@%%
Total@Abs@oct2List@%
```

```
Out[3550]= {e6, -1, -e7, -e3, -e5, e4, e2, -e3, e7, -1, e7, -1, -e1, -e6, e5, -e4, e1, -1, -e7, e2}
```

```
Out[3551]= 20
```

```
Out[3552]= 2 e2 - 2 e3 - 4
```

```
Out[3553]= 8
```

```
In[3554]:= (* 4-V =
```

$$15 \text{ element } \{e_1 \circ (e_2 \circ e_7), e_1 \circ (e_3 \circ e_7), e_1 \circ (e_4 \circ e_7), e_1 \circ (e_5 \circ e_7), e_1 \circ (e_6 \circ e_7), e_2 \circ (e_3 \circ e_7), e_2 \circ (e_4 \circ e_7), \\ e_2 \circ (e_5 \circ e_7), e_2 \circ (e_6 \circ e_7), e_3 \circ (e_4 \circ e_7), e_3 \circ (e_5 \circ e_7), e_3 \circ (e_6 \circ e_7), e_4 \circ (e_5 \circ e_7), e_4 \circ (e_6 \circ e_7), e_5 \circ (e_6 \circ e_7)\} *$$

```
In[3555]:= f@oct[{{#1}, #2}, 7] & /@ Subsets[Range@6, {2}]
Length@%
Total@%%
Total@Abs@oct2List@%
```

```
Out[3555]= {e5, -1, -e6, -e2, e4, -e4, e3, e1, -1, -e2, e6, -e5, -1, -e1, e3}
```

```
Out[3556]= 15
```

```
Out[3557]= -2 e2 + 2 e3 - 3
```

```
Out[3558]= 7
```

```
In[3559]:= (* 5-V = 6 element (without the factor of i as in the paper) -
            {e1°e7, e2°e7, e3°e7, e4°e7, e5°e7, e6°e7} *)
```

```
In[3560]:= (* i°**) f@oct[{{#}, 7] & /@ Range@6
Length@%
Total@%%
Total@Abs@oct2List@%
```

```
Out[3560]= {-e3, -e6, e1, -e5, e4, e2}
```

```
Out[3561]= 6
```

```
Out[3562]= e1 + e2 - e3 + e4 - e5 - e6
```

```
Out[3563]= 6
```

```
In[3564]:= (* 6-V = e1°(e2°(e3°(e4°(e5°e6)))) *)
```

```
In[3565]:= f@oct[Range@6]
Total@Abs@oct2List@%
```

```
Out[3565]= e7
```

```
Out[3566]= 1
```



```
In[3575]:= (* Mf reduced and sum of 0-V to 6-  
V is the same (without the factor of i as in the paper) *)  
% == %%%%%%%%%%
```

```
Out[3575]= True
```

```
In[3576]:= (* There are a total of 22 coefficients after octonion multiplication and combining *)  
Total@Abs@oct2List@%%
```

```
Out[3576]= 22
```

In[3854]:= (* The rewritten generating space, $a_{1,2,3}$ & $a_{1,2,3}^{*}$ basis *)

$$\text{In[3855]:= } \mathbf{a1} = \frac{1}{2} (-e_5 + \mathbf{i} \circ e_4)$$

$$\text{Out[3855]= } \frac{1}{2} (-e_2 - e_5)$$

In[3856]:= $\mathbf{a1}^*$

$$\text{Out[3856]= } \frac{e_2}{2} + \frac{e_5}{2}$$

$$\text{In[3857]:= } \mathbf{a2} = \frac{1}{2} (-e_3 + \mathbf{i} \circ e_1)$$

$$\text{Out[3857]= } \frac{1}{2} (-e_3 - 1)$$

In[3858]:= $\mathbf{a2}^*$

$$\text{Out[3858]= } \frac{e_3}{2} - \frac{1}{2}$$

$$\text{In[3859]:= } \mathbf{a3} = \frac{1}{2} (-e_6 + \mathbf{i} \circ e_2)$$

$$\text{Out[3859]= } \frac{1}{2} (e_4 - e_6)$$

In[3860]:= $\mathbf{a3}^*$

$$\text{Out[3860]= } \frac{e_6}{2} - \frac{e_4}{2}$$

In[3861]:= (* Commutation xoy *)

$$\text{In[3862]:= } \sum_{j=2}^6 \sum_{i=1}^{j-1} \mathbf{f@oct}[\{i, j\}] \circ \mathbf{f@oct}[\{j, i\}]$$

$$\text{Out[3862]= } 0$$

$$\text{In[3863]:= } \sum_{j=2}^6 \sum_{i=1}^{j-1} (\mathbf{f@oct}[\{i, j\}] + \mathbf{f@oct}[\{j, i\}])$$

$$\text{Out[3863]= } 0$$

In[3864]:= $\mathbf{f@}\{\mathbf{a1}, \mathbf{a2}\}$

$$\text{Out[3864]= } \frac{e_5}{2}$$

In[3865]:= $\mathbf{f@}\{\mathbf{a2}, \mathbf{a1}\}$

$$\text{Out[3865]= } \frac{e_2}{2}$$

In[3866]:= $\mathbf{f@}\{\mathbf{a1}, \mathbf{a2}\} + \mathbf{f@}\{\mathbf{a2}, \mathbf{a1}\}$

$$\text{Out[3866]= } \frac{e_2}{2} + \frac{e_5}{2}$$

In[3867]:= **f@{a1, a2} o f@{a2, a1}**

Out[3867]= $\frac{e_3}{2}$

In[3868]:= **f@{a1*, a2*}**

Out[3868]= $-\frac{e_2}{2}$

In[3869]:= **f@{a2*, a1*}**

Out[3869]= $-\frac{e_5}{2}$

In[3870]:= **f@{a1*, a2*} + f@{a2*, a1*}**

Out[3870]= $-\frac{e_2}{2} - \frac{e_5}{2}$

In[3871]:= **f@{a1*, a2*} o f@{a2*, a1*}**

Out[3871]= $-\frac{e_3}{2}$

In[3872]:= **f@{a1, a2*}**

Out[3872]= $\frac{e_2}{2}$

In[3873]:= **f@{a2*, a1}**

Out[3873]= $\frac{e_5}{2}$

In[3874]:= **f@{a1, a2*} + f@{a2*, a1}**

Out[3874]= $\frac{e_2}{2} + \frac{e_5}{2}$

In[3875]:= **f@{a1, a2*} o f@{a2*, a1}**

Out[3875]= $-\frac{e_3}{2}$

In[3876]:= (* Quarks and leptons as minimal left ideals *)

In[3877]:= $\omega = f@{\mathbf{a1}, \mathbf{a2}, \mathbf{a3}}$

$$\text{Out[3877]} = \frac{e_1}{4} - \frac{e_7}{4}$$

In[3878]:= ω^*

$$\text{Out[3878]} = \frac{e_7}{4} - \frac{e_1}{4}$$

In[3879]:= $\mathbf{vc} = \omega \circ \omega^*$

$$\text{Out[3879]} = \frac{1}{8}$$

In[3880]:= \mathbf{vc}^*

$$\text{Out[3880]} = \frac{1}{8}$$

In[3881]:= $\mathbf{a2}^* \circ \mathbf{a1} == f@{\mathbf{a2}^*, \mathbf{a1}}$

Out[3881]= True

In[3882]:= $f@{\mathbf{a2}^*, \mathbf{a1}}$

$$\text{Out[3882]} = \frac{e_5}{2}$$

In[3883]:= $\mathbf{a1}^* \circ \mathbf{a2}$

$$\text{Out[3883]} = -\frac{e_5}{2}$$

In[3884]:= $\Delta 1 = -\mathbf{a2}^* \circ \mathbf{a1} - \mathbf{a1}^* \circ \mathbf{a2}$

Out[3884]= 0

In[3885]:= $\Delta 2 = -f@{\mathbf{i}, \mathbf{a2}^*, \mathbf{a1}} - f@{\mathbf{i}, \mathbf{a1}^*, \mathbf{a2}}$

Out[3885]= 0

In[3886]:= $\mathbf{a1}^* \circ \mathbf{a1} == \mathbf{a2}^* \circ \mathbf{a2} == \mathbf{a3}^* \circ \mathbf{a3} == \frac{1}{2}$

Out[3886]= True

In[3887]:= $\Delta 3 = \mathbf{a2}^* \circ \mathbf{a2} - \mathbf{a1}^* \circ \mathbf{a1}$

Out[3887]= 0

In[3888]:= $\Delta 4 = -\mathbf{a1}^* \circ \mathbf{a3} - \mathbf{a3}^* \circ \mathbf{a1}$

Out[3888]= 0

In[3889]:= $\Delta 4 = -\mathbf{a1}^* \circ \mathbf{a3} - \mathbf{a3}^* \circ \mathbf{a1}$

Out[3889]= 0

In[3890]:= $\Delta 5 = -a1^* \circ a3 + a3^* \circ a1$

Out[3890]= e_7

In[3891]:= $\Delta 5 = -f@{\mathbf{i}, a1^* a3} + f@{\mathbf{i}, a3^* a1}$
FullSimplify@%

Out[3891]= $\frac{1}{4} e_5 (e_2 + e_5) - \frac{1}{2} \left(\frac{e_2}{2} + \frac{e_5}{2} \right) e_5$

Out[3892]= 0

In[3893]:= $\Delta 6 = -a3^* \circ a2 - a2^* \circ a3$

Out[3893]= 0

In[3894]:= $\Delta 7 = f@{\mathbf{i}, a3^*, a2} - f@{\mathbf{i}, a2^*, a3}$

Out[3894]= e_5

In[3895]:= $\Delta 8 = -\frac{1}{\sqrt{3}} (a1^* \circ a1 - a2^* \circ a2 - 2 a3^* \circ a3)$

Out[3895]= $\frac{1}{\sqrt{3}}$

In[3924]:= (* Quarks and leptons as minimal left ideals *)

$$\text{In[3925]:= } \mathcal{S}u = \left(\mathcal{V} + \overline{\mathcal{D}r} a1^* + \overline{\mathcal{D}g} a2^* + \overline{\mathcal{D}b} a3^* + \right. \\ \left. \mathcal{U}r a3^* a2^* + \mathcal{U}g a1^* a3^* + \mathcal{U}b a2^* a1^* + \right. \\ \left. \overline{\mathcal{E}} a3^* a2^* a1^* \right) \mathcal{V}c$$

FullSimplify@Su

$$\text{Out[3925]= } \frac{1}{8} \left(\left(\frac{e_6}{2} - \frac{e_4}{2} \right) \overline{\mathcal{D}b} + \left(\frac{e_3}{2} - \frac{1}{2} \right) \overline{\mathcal{D}g} + \left(\frac{e_2}{2} + \frac{e_5}{2} \right) \overline{\mathcal{D}r} + \left(\frac{e_3}{2} - \frac{1}{2} \right) \left(\frac{e_2}{2} + \frac{e_5}{2} \right) \left(\frac{e_6}{2} - \frac{e_4}{2} \right) \overline{\mathcal{E}} + \right. \\ \left. \mathcal{V} + \mathcal{U}b \left(\frac{e_3}{2} - \frac{1}{2} \right) \left(\frac{e_2}{2} + \frac{e_5}{2} \right) + \mathcal{U}g \left(\frac{e_2}{2} + \frac{e_5}{2} \right) \left(\frac{e_6}{2} - \frac{e_4}{2} \right) + \mathcal{U}r \left(\frac{e_3}{2} - \frac{1}{2} \right) \left(\frac{e_6}{2} - \frac{e_4}{2} \right) \right)$$

$$\text{Out[3926]= } \frac{1}{64} \left(4(e_6 - e_4) \overline{\mathcal{D}b} + 4(e_3 - 1) \overline{\mathcal{D}g} + 4(e_2 + e_5) \overline{\mathcal{D}r} - (e_3 - 1)(e_2 + e_5)(e_4 - e_6) \overline{\mathcal{E}} + \right. \\ \left. 8\mathcal{V} + 2\mathcal{U}b(e_3 - 1)(e_2 + e_5) - 2\mathcal{U}g(e_2 + e_5)(e_4 - e_6) - 2\mathcal{U}r(e_3 - 1)(e_4 - e_6) \right)$$

$$\text{In[3927]:= } \mathcal{S}d = \left(\overline{\mathcal{V}} - \mathcal{D}r a1 - \mathcal{D}g a2 - \mathcal{D}b a3 + \right. \\ \left. \overline{\mathcal{U}r} a3 a2 + \overline{\mathcal{U}g} a1 a3 + \overline{\mathcal{U}b} a2 a1 + \right. \\ \left. \overline{\mathcal{E}} a1 a2 a3 \right) \mathcal{V}c^*$$

FullSimplify@Sd

$$\text{Out[3927]= } \frac{1}{8} \left(\overline{\mathcal{V}} + \frac{1}{4}(-e_3 - 1)(-e_2 - e_5) \overline{\mathcal{U}b} + \frac{1}{4}(-e_2 - e_5)(e_4 - e_6) \overline{\mathcal{U}g} + \frac{1}{4}(-e_3 - 1)(e_4 - e_6) \overline{\mathcal{U}r} - \right. \\ \left. \frac{1}{2} \mathcal{D}b(e_4 - e_6) - \frac{1}{2} \mathcal{D}g(-e_3 - 1) - \frac{1}{2} \mathcal{D}r(-e_2 - e_5) + \frac{1}{8}(-e_3 - 1)(-e_2 - e_5)(e_4 - e_6) \overline{\mathcal{E}} \right)$$

$$\text{Out[3928]= } \frac{1}{64} \left(8\overline{\mathcal{V}} + 2(e_3 + 1)(e_2 + e_5) \overline{\mathcal{U}b} - 2(e_2 + e_5)(e_4 - e_6) \overline{\mathcal{U}g} - \right. \\ \left. 2(e_3 + 1)(e_4 - e_6) \overline{\mathcal{U}r} + 4\mathcal{D}b(e_6 - e_4) + 4\mathcal{D}g(e_3 + 1) + 4\mathcal{D}r(e_2 + e_5) + (e_3 + 1)(e_2 + e_5)(e_4 - e_6) \overline{\mathcal{E}} \right)$$

In[3929]:= FullSimplify@Su* == FullSimplify@Sd

$$\text{Out[3929]= } \frac{1}{64} \left(e_4 \left(4\overline{\mathcal{D}b} + (e_2 + e_5) \left((e_3 - 1) \overline{\mathcal{E}} + 2\mathcal{U}g \right) - 2(\mathcal{U}r + 4\mathcal{V}) + 2\mathcal{U}r e_3 \right) - \right. \\ \left. e_6 \left(4\overline{\mathcal{D}b} + (e_2 + e_5) \left((e_3 - 1) \overline{\mathcal{E}} + 2\mathcal{U}g \right) - 2\mathcal{U}r + 8\mathcal{V} + 2\mathcal{U}r e_3 \right) - 4e_3 \left(\overline{\mathcal{D}g} + 2\mathcal{V} \right) - 4\overline{\mathcal{D}g} - \right. \\ \left. 2e_2 \left(2\overline{\mathcal{D}r} - \mathcal{U}b + 4\mathcal{V} + \mathcal{U}b e_3 \right) - 2e_5 \left(2\overline{\mathcal{D}r} - \mathcal{U}b + 4\mathcal{V} + \mathcal{U}b e_3 \right) + 8\mathcal{V} - 8\mathcal{V} e_1 - 8\mathcal{V} e_7 \right) = \\ \frac{1}{64} \left(8\overline{\mathcal{V}} + 2(e_3 + 1)(e_2 + e_5) \overline{\mathcal{U}b} - 2(e_2 + e_5)(e_4 - e_6) \overline{\mathcal{U}g} - 2(e_3 + 1)(e_4 - e_6) \overline{\mathcal{U}r} + \right. \\ \left. 4\mathcal{D}b(e_6 - e_4) + 4\mathcal{D}g(e_3 + 1) + 4\mathcal{D}r(e_2 + e_5) + (e_3 + 1)(e_2 + e_5)(e_4 - e_6) \overline{\mathcal{E}} \right)$$

In[3936]:= $\tau 1 = \omega + \omega^*$

Out[3936]= 0

In[3937]:= $\tau 2 = \mathbf{i} \circ \omega - \mathbf{i} \circ \omega^*$

$$\text{Out[3937]= } \frac{e_3}{2} - \frac{1}{2}$$

In[3938]:= $\tau 3 = \omega \circ \omega^* - \omega^* \circ \omega$

Out[3938]= 0