3D Polytope Hulls of $E_8$ 421, 241, and 142

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Using rows 2 through 4 of a unimodular $8 \times 8$ rotation matrix, the vertices of $E_8$ 421, 241, and 142 are projected to 3D and then gathered & tallied into groups by the norm of their projected locations. The resulting Platonic and Archimedean solid 3D structures are then used to study $E_8$’s relationship to other research areas, such as sphere packings in Grassmannian spaces, using $E_8$ Eisenstein Theta Series in recent proofs for optimal 8D and 24D sphere packings, nested lattices, and quantum basis critical parity proofs of the Bell-Kochen-Specker (BKS) theorem.

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I. INTRODUCTION

This paper will introduce several possible new connections between $E_8$ 421, 241, and 142 and the study of sphere packings in Grassmannian spaces[1], using $E_8$ Eisenstein Theta Series in recent proofs for optimal 8D and 24D sphere packings[2], nested lattices[3], and quantum basis critical parity proofs of the Bell-Kochen-Specker (BKS) theorem[4].

A. Generating Polytopes

Fig. 1 is the Petrie projection of the largest of the exceptional simple Lie algebras, groups and lattices called $E_8$. The Split Real Even (SRE) form of $E_8$ has a 421 Gosset polytope of 240 vertices and 6720 edges of 8 dimensional (8D) length $\sqrt{2}$. In addition to this $E_8$ root polytope identified with the Dynkin diagram shown in Fig. 2a, there are $2^8 - 1 = 255$ possible permutations of the $E_8$ Dynkin diagram. Several of these other permutations are commonly represented visually using the Petrie projection basis. Among these others are the 2,160 vertex 241 and the 17,280 vertex 142 polytope, which are constructed by generating the resulting roots by moving the ringed (or filled) node to the other ends of the Dynkin diagram, as shown in Figs. 2b and 2c respectively.

Interestingly, $E_8$ has been shown[5] to fold to the 4D polychora of $H_4$ (aka. the 120 vertex 720 edge 600-cell) and a scaled copy $H_4 \Phi$, where $\Phi = \frac{1}{2} (1 + \sqrt{5}) = 1.618...$ is the big golden ratio and $\varphi = \frac{1}{2} (\sqrt{5} - 1) = 1/\Phi = \Phi - 1 = 0.618...$ is the small golden ratio.

FIG. 1: $E_8$ 421 Petrie projection

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FIG. 2: $E_8$ Dynkin diagrams a) 421, b) 241, c) 142

Also shown are the Cartan and simple root matrices which correspond to the common Coxeter-Dynkin representation of the diagrams.
B. 8D Platonic Rotation

In a previous paper[6], a unimodular form of a specific matrix for performing an 8D rotation of the SRE $E_8$ group of root vertices results in the vertices of $H_4$ (a.k.a. the 600-cell). This rotation (or folding) matrix is related to the Platonic solid icosahedron and was shown to be that of (1).

$$H_{4\text{uni}} = \begin{pmatrix} \sqrt{\phi^2} & 0 & 0 & 0 & 1/\sqrt{\phi} & 0 & 0 & 0 \\ 0 & -\sqrt{\phi} & 1/\sqrt{\phi} & 0 & 0 & \sqrt{\phi} & 1/\sqrt{\phi} & 0 \\ 0 & 1/\sqrt{\phi} & 0 & -\sqrt{\phi} & 0 & 1/\sqrt{\phi} & 0 & \sqrt{\phi} \\ 0 & 0 & -\sqrt{\phi} & 1/\sqrt{\phi} & 0 & 0 & \sqrt{\phi} & 1/\sqrt{\phi} \\ 0 & \sqrt{\phi} & 1/\sqrt{\phi} & 0 & 0 & -\sqrt{\phi} & 1/\sqrt{\phi} & 0 \\ 0 & 0 & -\sqrt{\phi} & 1/\sqrt{\phi} & 0 & 0 & -\sqrt{\phi} & 1/\sqrt{\phi} \end{pmatrix} /2$$

(1)

More interestingly from [7], $H_{4\text{uni}}$ can be generated using a combination of the unitary Hermitian matrices commonly used for Quantum Computing (QC) qubit logic, namely those of the 2 qubit CNOT (2) and SWAP (3) gates. Taking these patterns, combined with the recursive functions that build $\Phi$ from the Fibonacci sequence, it is straightforward to derive $H_{4\text{uni}}$ from scaled QC logic gates. $H_{4\text{uni}}$ is shown in Fig. 3.

CNOT= $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

SWAP= $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

D. 3D Platonic Solid Projection Prism

While 3D projections can be generated for each set of basis vectors used in Appendix B, there are only a few that render interesting 3D structures. A few of these are presented in Appendix C Figs. 12-13.

The most interesting 3D projections are found in Figs. 14-18 in Appendix C showing various projections of $4_{21}$, $2_{41}$, and $1_{42}$ which are based on “E8→H4”. This basis is derived from the Platonic solid icosahedron. The twelve vertices of the icosahedron can be decomposed into three mutually-perpendicular golden rectangles (as shown in Fig. 4), whose boundaries are linked in the pattern of the Borromean rings. Rows (or columns) 2-4 (or 5-8) of $H_{4uni}$ contain 6 of the 12 vertices of this icosahedron, including 2 at the origin with the other 6 of 12 icosahedron vertices being the antipodal reflection of these through the origin. These 2 (or 3) rows are then used as a kind of “Platonic solid projection prism” to form the 2 (or 3) 8D basis vectors used in the 2D (or 3D) projection $4_{21}$, $2_{41}$, and $1_{42}$.
FIG. 4: The icosahedron formed from 3 mutually-perpendicular golden rectangles

\[
\begin{align*}
X &= \{0 \ -\sqrt{\frac{2}{\sqrt{5}+3}} \ \sqrt{\frac{2}{\sqrt{5}+3}} \ 0 \ 0 \ \sqrt{\frac{2}{\sqrt{5}+3}} \ \sqrt{\frac{2}{\sqrt{5}+3}} \ 0 \} \\
Y &= \{0 \ \sqrt{\frac{2}{\sqrt{5}+1}} \ 0 \ -\sqrt{\frac{2}{\sqrt{5}+3}} \ 0 \ \sqrt{\frac{2}{\sqrt{5}+3}} \ 0 \} \\
Z &= \{0 \ 0 \ -\sqrt{\frac{2}{\sqrt{5}+3}} \ \sqrt{\frac{2}{\sqrt{5}+3}} \ 0 \ \sqrt{\frac{2}{\sqrt{5}+3}} \ \sqrt{\frac{2}{\sqrt{5}+3}} \} \\
\end{align*}
\]

(6)

This Platonic solid projection in 3D manifests a large number of concentric hulls with Platonic and Archimedean solid related structures. The 8 hull \(4_{21}\), which includes two 4 hull 600-cell structures \((H_4 \& H_4\Phi)\), is shown in Fig. 5. The much larger sets of \(2_{41}\) and \(1_{42}\) are shown in Appendix C Figs. 14-16.

For example, the 3rd largest of 74 hulls in \(1_{42}\) is a pair of overlapping 60 vertex rhombicosidodecahedrons shown in more detail here in Fig. 6. It is an Archimedean solid, one of thirteen convex isogonal nonprismatic solids. The 4th largest hull is a 120 vertex non-uniform truncated icosidodecahedron shown in Fig. 7.

II. PLATONIC LINK TO SPHERE PACKINGS

Fig. 5 of [1] shows a geometric structure with 110 antipodal points from the union of the vertex sets of a dodecahedron (20), an icosidodecahedron (30), and a truncated icosahedron (60). This is structurally the same as that shown in Fig. 18c & 18d without the icosahedron that centers on the pentagons of the truncated icosahedron. The other difference is in the fact that the truncation of the icosahedron in the \(E_8\) projection is not regular and results in a non-uniform rhombicosidodecahedron as shown in Figs. 17a and 18b.

These figures use only the outer two hulls of \(4_{21}\), \(2_{41}\), and \(1_{42}\). The smaller of each of these three pairs of hulls are scaled up to unit norm. For \(4_{21} = 2_{41}\), the scale factor used on the overlapping pair of icosahedrons is

\[
\sqrt{\frac{8}{(5 + \sqrt{5})}} \approx 1.0514. \quad \text{For} \ 1_{42}, \quad \text{the scale factor used on the overlapping pair of dodecahedrons is} \quad \sqrt{\frac{16+32\sqrt{5}}{15+33\sqrt{5}}} \approx 1.0092.
\]

It would be interesting to calculate the sphere packing efficiencies of the geometries shown in Fig. 18 as well as those individual hulls shown in Figs. 14-15. While it is straightforward to calculate the projected vertex positions given the information in this paper, these are available in a Mathematica notebook that is available on the author’s website http://www.TheoryOfEverything.org/TOE/JGM/3D-Polytope-Hulls-of-E8.nb.

Also related to sphere packings, it was recently proven[2] that the \(E_8\) root lattice and the Leech lattice are universally optimal among point configurations in Euclidean spaces of dimensions 8 and 24, respectively. The proof relies on the use Laplace transforms of quasimodular forms related to the Eisenstein \(E_4(q)\) Series integers that are the Theta series of the \(E_8\) lattice. This series from http://oeis.org/A004009 is
III. PLATONIC LINK TO NESTED LATTICES

Also related to $E_8$’s Gossett and Witting polytopes and the aforementioned $4_{21}$ and $2_{41}$ Dynkin diagram permutations, a blog article on nested lattices[3] credits Warren D. Smith with observing that the sum of the first three terms in the Theta Series of $E_8$ is a perfect fourth power $1 + 240 + 2160 = 240^4 = 7^3$. The vertices of the related $E_8$ groups in this series $4_{21}$ and $2_{41}$ have been visualized in Figs. 9-10 of Appendix B. The 2160 vertex $2_{41}$ is also visualized in 3D in Figs. 12 and 14.

IV. PLATONIC LINK TO BELL-KOCHEN-SPECKER (BKS) PARITY PROOFS

The 4D 120 vertex 600-cell ($H_4$) has been shown[5] to be easily generated by using $H_{4uni}$ in (1) to rotate $E_8 \rightarrow H_4$. The dual 3D Platonic solid structure of icosahedrons and dodecahedrons (embedded in the rhombic triacontahedron) are contained within the 120 vertex 4D 600-cell, itself a combination of the self-dual 24-cells (i.e., 8-cell aka. the tesseract or hyper-cube and the 16-cell orthoplex or cross polytope). Indeed, it has been shown[8] that the 3D Platonic solid structures can be a generator of $H_4$.

In $\mathbb{C}P^3$, the Penrose dodecahedron of the BKS theorem and the Witting polytope, which is the Complex 4D representation of the SRE $E_8$ used herein, are shown to be identical[9]. This same structure has been linked to the Bell-Kochen-Specker (BKS) parity proofs[4].

It would be interesting to calculate the proofs from the geometries shown in Fig. 18 as well as those individual hulls shown in Figs. 14-15.

V. CONCLUSION

This paper has introduced new visualizations and connections between $E_8$, $4_{21}$, $2_{41}$, and $1_{42}$ and the study of sphere packings in Grassmannian spaces[1], sphere packing proofs using $E_8$ Eisenstein Theta Series for optimal packing in 8D and 24D, nested lattices, and quantum basis critical parity proofs of the Bell-Kochen-Specker (BKS) theorem. It is anticipated that these visualizations and connections will be useful in discovering new insights into unifying the mathematical symmetries as related to unification in theoretical physics.

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I would like to thank my wife for her love and patience and those in academia who have taken the time to review this work.
Appendix A: Mathematica\textsuperscript{TM} code to generate the vertex sets 4_{21}, 2_{41}, and 1_{42}

```mathematica
(* Permutation functions *)
perm4[n_] := Flatten[Outer[List, Sequence @@ Table[{-1, 1}, {n}]], n - 1];
perms8[a_, b_, c_, d_, e_, f_, g_, h_] := Flatten[Permutations@{a b c d e f g h}, 1];
perms8@in_ := Select[perms8@in, EvenQ@Count[Sign@#, -1] &];

(* E8 1,42 vertices *)
e8142 = Union@Join[perms8@{4, 2, 2, 2, 2, 0, 0, 0},
  Flatten[perms8@a & /@ {
    {2, 2, 2, 2, 2, 2, 2, 2},
    {5, 1, 1, 1, 1, 1, 1, 1},
    {3, 3, 3, 1, 1, 1, 1, 1}}]/4;
Length@%
17280

(* Even to Odd permutations in the last digit *)
e8142 = If[Total@Abs@e8142[[n]] == 3, e8142[[n]], ReplacePart[e8142[[n]], 8 -> -e8142[[n], 8]]] & /@ Range@Length@e8142;

(* E8 2,41 vertices *)
e8241 = Union@Join[
  perms8[{1, 0, 0, 0, 0, 0, 0, 4}],
  perms8[{1, 1, 1, 1, 0, 0, 0, 2}],
  Eperms8[2, 0, 0, 0, 0, 0, 0, 0 + 1)]]/4;
Length@%
2160

(* Even to Odd permutations in the last digit *)
e8241 = (**) \sqrt{2} (**) If[Total@Abs@e8241[[n]] > 2, ReplacePart[e8241[[n]], 8 -> -e8241[[n], 8]], e8241[[n]]] & /@ Range@Length@e8241;

(* E8 4,21 vertices scaled for Max Norm=1 from Unimodular C600 and it's 3D E8-H4 projection basis *)
e8421 = (**) 8 (**) Union@Join[Eperms8@{1, 1, 1, 1, 1, 1, 1, 1} / 2, perms8@{1, 1, 0, 0, 0, 0, 0, 0}] (**) \sqrt{2} (**);
Length@%
240

FIG. 8: Mathematica\textsuperscript{TM} code to generate the vertex sets 4_{21}, 2_{41}, and 1_{42}

Appendix B: Various 2D projections of 4_{21}, 2_{41}, and 1_{42} Figs. 9-11

Appendix C: Various 3D projections of 4_{21}, 2_{41}, and 1_{42} Figs. 12-18
FIG. 9: $4_{21}$ Polytope projected to various planes
Each 2D projection shown lists the projection name, the numeric basis vectors used, and the $4_{21}$ overlap color coded vertex groups, and the projection with vertices & 6720 edges
FIG. 10: $4_{21}$ Polytope projected to various planes
Each 2D projection shown lists the projection name, the numeric basis vectors used, and the $4_{21}$ & $2_{41}$ overlap color coded vertex groups, and the projection with vertices (larger) & 6720 edges and the $2_{41}$ vertices (smaller)
FIG. 11: $4_{21}$ Polytope projected to various planes
Each 2D projection shown lists the projection name, the numeric basis vectors used, and the $4_{21}$ & $1_{42}$ overlap color coded vertex groups, and the projection with vertices (larger) & 6720 edges and the $1_{42}$ vertices (smaller)
FIG. 12: $4_21$ & Polytope projected to various 3D spaces
Each 3D projection shown lists the projection name, the numeric basis vectors used, and the $4_21$ & $2_41$ overlap color coded vertex groups, and the projection with vertices (larger) & 6720 edges and the $2_41$ vertices (smaller)

FIG. 13: $4_21$ & Polytope projected to various 3D spaces
Each 3D projection shown lists the projection name, the numeric basis vectors used, and the $4_21$ & $1_{42}$ overlap color coded vertex groups, and the projection with vertices (larger) & 6720 edges and the $1_{42}$ vertices (smaller)
FIG. 14: Concentric hulls of $2_4$ in Platonic 3D projection with vertex count in each hull and increasing opacity and varied surface colors.  

a) 24 individual concentric hulls 

b) In groups of 8 hulls
FIG. 15: Concentric hulls of $1_{42}$ in Platonic 3D projection with vertex count in each hull and increasing opacity and varied surface colors.

a) 74 individual concentric hulls
b) In groups of 8 hulls
FIG. 16: Combined concentric hulls of $2_{41}$ and $1_{42}$ in Platonic 3D projection with increasing opacity and varied surface colors. Also listing grouped vertex counts color coded by overlaps (black text) and norm distances and vertex counts (red text).

a) 24 hulls of $2_{41}$
b) 74 hulls of $1_{42}$
FIG. 17: \( E_8 \)'s outer two hulls scaled to unit norms in Platonic 3D projection with increasing opacity and varied surface colors. Also listing grouped vertex counts color coded by overlaps (black text) and norm distances and vertex counts (red text).

a) 100 vertex \( 1_{42} \) non-uniform rhombicosidodecahedron (60 yellow vertices) & two overlapping dodecahedrons (20 red vertices)

b) 208 vertex combination of a, adding two sets of \( 4_{21} = 2_{41} \) icosidodecahedrons (30 red) & four overlapping icosahedrons (12 cyan vertices)

Note: The internal numbers of the image are the 8 axis (i.e. the projection basis vectors).
FIG. 18: $E_{24}$'s outer two hulls scaled to unit norms in Platonic 3D projection with vertex counts color coded by overlaps

a) 54 vertex (42 unique) $4_{21} = 2_{41}$ icosidodecahedron (30 yellow) & two overlapping icosahedrons (12 red) scaled 1.051

b) 100 vertex (80 unique) $1_{42}$ non-uniform rhombicosidodecahedron (60 yellow) & two overlapping dodecahedrons (20 red) scaled 1.0092

c) 154 vertex (122 unique) combination of a & b

d) 208 vertex (122 unique) combination same as c with color coded vertex counts for both $4_{21}$ & $2_{41}$

Note: The internal numbers of the image are the 8 axis (projection basis vectors).