Three historical (reverse order) forms of the E8 ↔ H_4 folding matrix :

The current centrosymmetric, traceless (Tr=0), and volume preserving (Det = 1) form with features analyzed in:

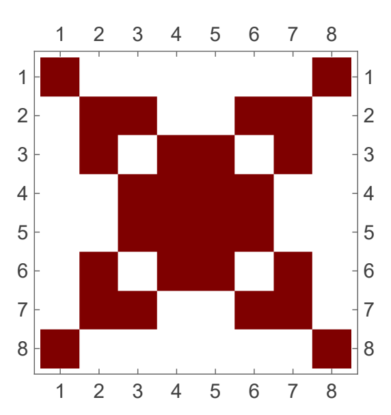
- The_Isomorphism_of_H4_and_E8 https://arxiv.org/abs/2311.01486
- The_Isomorphism_of_3-Qubit_Hadamards_and_E8 https://arxiv.org/abs/2311.11918

It transforms a (Left 4D) H_4 (the 120 vertex 600-cell) into a second scaled (palindromically reversed Right 4D) φH_4 copy with a Galois transform $\varphi \leftrightarrow -1/\varphi$.
Those L+R (now 8D) are palindromically reversed and combined into 240 vertices by $H_4^2 = H_{4\,L+R} + \varphi H_{4\,L+R}$ that give you back E_8 upon rotation (folding/unfolding) by U^{-1}.

```
In[•]:= matCheck@octSym@Det1𝕌 // MatrixForm
```

Out[•]//MatrixForm=

$$\begin{pmatrix}
-\frac{1}{2\varphi^{3/2}} & 0. & 0. & 0. & 0. & 0. & 0. & -\frac{\varphi^{3/2}}{2} \\
0. & -\frac{1}{2\sqrt{\varphi}} & \frac{\sqrt{\varphi}}{2} & 0. & 0. & \frac{\sqrt{\varphi}}{2} & \frac{1}{2\sqrt{\varphi}} & 0. \\
0. & \frac{\sqrt{\varphi}}{2} & 0. & -\frac{1}{2\sqrt{\varphi}} & \frac{1}{2\sqrt{\varphi}} & 0. & \frac{\sqrt{\varphi}}{2} & 0. \\
0. & 0. & -\frac{1}{2\sqrt{\varphi}} & \frac{\sqrt{\varphi}}{2} & \frac{\sqrt{\varphi}}{2} & \frac{1}{2\sqrt{\varphi}} & 0. & 0. \\
0. & 0. & \frac{1}{2\sqrt{\varphi}} & \frac{\sqrt{\varphi}}{2} & \frac{\sqrt{\varphi}}{2} & -\frac{1}{2\sqrt{\varphi}} & 0. & 0. \\
0. & \frac{\sqrt{\varphi}}{2} & 0. & \frac{1}{2\sqrt{\varphi}} & -\frac{1}{2\sqrt{\varphi}} & 0. & \frac{\sqrt{\varphi}}{2} & 0. \\
0. & \frac{1}{2\sqrt{\varphi}} & \frac{\sqrt{\varphi}}{2} & 0. & 0. & \frac{\sqrt{\varphi}}{2} & -\frac{1}{2\sqrt{\varphi}} & 0. \\
-\frac{\varphi^{3/2}}{2} & 0. & 0. & 0. & 0. & 0. & 0. & -\frac{1}{2\varphi^{3/2}}
\end{pmatrix}$$



Matrix Types:

- ☑ Is a Matrix
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- ☐ Diagonal
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- ☑ CentroSymmetric[AJ = JA]
- ☐ Hamiltonian[J=A$^{T\text{ or }\dagger}$JA]
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- ☑ SkewCentroSymmetric[AJ=−JA]
- ☐ SkewHamiltonian[J⇒−J]

```
In[•]:= inN@Det@Det1𝕌
```

Out[•]=

$$\boxed{\begin{matrix}\frac{1}{256}\left(-\frac{64}{\varphi^9}+\frac{64}{\varphi^3}\right)\varphi^6 \\ 1.\end{matrix}}$$

Det1𝕌 = Det1𝕌$^{-1}$ by swapping (non − outside corner) $\sqrt{\varphi} \leftrightarrow \frac{1}{\sqrt{\varphi}}$

```
In[•]:= octSym@Inverse@Det1𝕌 // MatrixForm
```

Out[•]//MatrixForm=

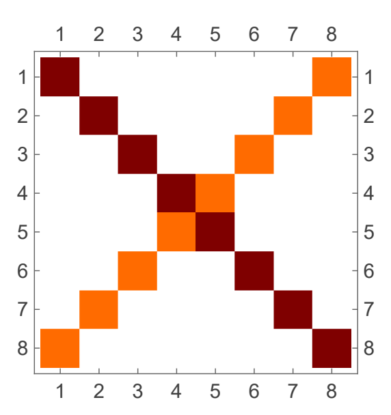
$$\begin{pmatrix}
\frac{1}{2\varphi^{3/2}} & 0. & 0. & 0. & 0. & 0. & 0. & -\frac{\varphi^{3/2}}{2} \\
0. & -\frac{\sqrt{\varphi}}{2} & \frac{1}{2\sqrt{\varphi}} & 0. & 0. & \frac{1}{2\sqrt{\varphi}} & \frac{\sqrt{\varphi}}{2} & 0. \\
0. & \frac{1}{2\sqrt{\varphi}} & 0. & -\frac{\sqrt{\varphi}}{2} & \frac{\sqrt{\varphi}}{2} & 0. & \frac{1}{2\sqrt{\varphi}} & 0. \\
0. & 0. & -\frac{\sqrt{\varphi}}{2} & \frac{1}{2\sqrt{\varphi}} & \frac{1}{2\sqrt{\varphi}} & \frac{\sqrt{\varphi}}{2} & 0. & 0. \\
0. & 0. & \frac{\sqrt{\varphi}}{2} & \frac{1}{2\sqrt{\varphi}} & \frac{1}{2\sqrt{\varphi}} & -\frac{\sqrt{\varphi}}{2} & 0. & 0. \\
0. & \frac{1}{2\sqrt{\varphi}} & 0. & \frac{\sqrt{\varphi}}{2} & -\frac{\sqrt{\varphi}}{2} & 0. & \frac{1}{2\sqrt{\varphi}} & 0. \\
0. & \frac{\sqrt{\varphi}}{2} & \frac{1}{2\sqrt{\varphi}} & 0. & 0. & \frac{1}{2\sqrt{\varphi}} & -\frac{\sqrt{\varphi}}{2} & 0. \\
-\frac{\varphi^{3/2}}{2} & 0. & 0. & 0. & 0. & 0. & 0. & \frac{1}{2\varphi^{3/2}}
\end{pmatrix}$$

Det1𝕌^2 as the Dot product forms diagonal Cartan-like matrix of this folding matrix Det1𝕌.Det1𝕌:

```
In[•]:= matCheck@octSym[Det1𝕌.Det1𝕌] // MatrixForm
```

Out[•]//MatrixForm=

$$\begin{pmatrix}
\frac{1}{4}\left(\frac{1}{\varphi^3}+\varphi^3\right) & 0. & 0. & 0. & 0. & 0. & 0. & \frac{1}{2} \\
0. & \frac{1}{4}\left(\frac{1}{\varphi^3}+\varphi^3\right) & 0. & 0. & 0. & 0. & \frac{1}{2} & 0. \\
0. & 0. & \frac{1}{4}\left(\frac{1}{\varphi^3}+\varphi^3\right) & 0. & 0. & \frac{1}{2} & 0. & 0. \\
0. & 0. & 0. & \frac{1}{4}\left(\frac{1}{\varphi^3}+\varphi^3\right) & \frac{1}{2} & 0. & 0. & 0. \\
0. & 0. & 0. & \frac{1}{2} & \frac{1}{4}\left(\frac{1}{\varphi^3}+\varphi^3\right) & 0. & 0. & 0. \\
0. & 0. & \frac{1}{2} & 0. & 0. & \frac{1}{4}\left(\frac{1}{\varphi^3}+\varphi^3\right) & 0. & 0. \\
0. & \frac{1}{2} & 0. & 0. & 0. & 0. & \frac{1}{4}\left(\frac{1}{\varphi^3}+\varphi^3\right) & 0. \\
\frac{1}{2} & 0. & 0. & 0. & 0. & 0. & 0. & \frac{1}{4}\left(\frac{1}{\varphi^3}+\varphi^3\right)
\end{pmatrix}$$



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- ☐ SkewHamiltonian[J⇒−J]

An older (but easier to read symbolically) Det≠1 form:

This is just an unscaled version of the current Det1𝕌...

```
In[•]:= 𝕌Det1f
        Det1𝕌 == 𝕌 / % // MatrixForm
```

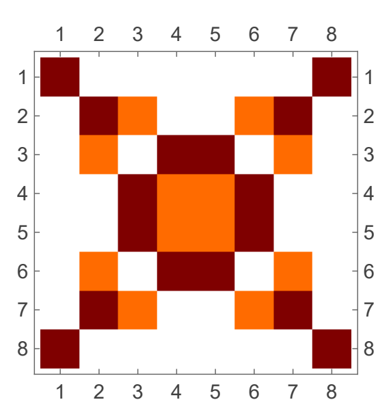
Out[•]= $2\sqrt{\frac{1}{\varphi}}$

Out[•]//MatrixForm=
True

```
In[•]:= matCheck@octSym@𝕌 // MatrixForm
```

Out[•]//MatrixForm=

$$\begin{pmatrix}
-\frac{1}{\varphi^2} & 0. & 0. & 0. & 0. & 0. & 0. & -\varphi \\
0. & -\frac{1}{\varphi} & 1 & 0. & 0. & 1 & \frac{1}{\varphi} & 0. \\
0. & 1 & 0. & -\frac{1}{\varphi} & \frac{1}{\varphi} & 0. & 1 & 0. \\
0. & 0. & -\frac{1}{\varphi} & 1 & 1 & \frac{1}{\varphi} & 0. & 0. \\
0. & 0. & \frac{1}{\varphi} & 1 & 1 & -\frac{1}{\varphi} & 0. & 0. \\
0. & 1 & 0. & \frac{1}{\varphi} & -\frac{1}{\varphi} & 0. & 1 & 0. \\
0. & \frac{1}{\varphi} & 1 & 0. & 0. & 1 & -\frac{1}{\varphi} & 0. \\
-\varphi & 0. & 0. & 0. & 0. & 0. & 0. & -\frac{1}{\varphi^2}
\end{pmatrix}$$



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```
In[•]:= inN@Det@𝕌
```

Out[•]=

$$\boxed{\begin{matrix}\frac{-64\varphi^3+64\varphi^9}{\varphi^{10}} \\ 37.3499\end{matrix}}$$

```
In[•]:= octSym@Inverse@𝕌 // MatrixForm
```

Out[•]//MatrixForm=

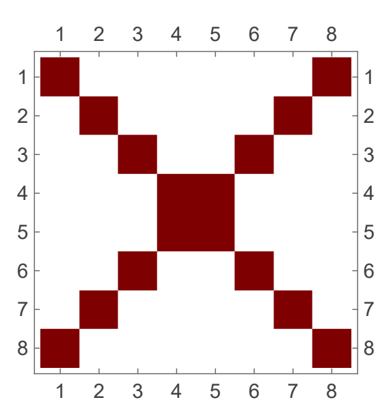
$$\begin{pmatrix}
\frac{1}{4\varphi} & 0. & 0. & 0. & 0. & 0. & 0. & -\frac{\varphi^2}{4} \\
0. & -\frac{\varphi}{4} & \frac{1}{4} & 0. & 0. & \frac{1}{4} & \frac{\varphi}{4} & 0. \\
0. & \frac{1}{4} & 0. & -\frac{\varphi}{4} & \frac{\varphi}{4} & 0. & \frac{1}{4} & 0. \\
0. & 0. & -\frac{\varphi}{4} & \frac{1}{4} & \frac{1}{4} & \frac{\varphi}{4} & 0. & 0. \\
0. & 0. & \frac{\varphi}{4} & \frac{1}{4} & \frac{1}{4} & -\frac{\varphi}{4} & 0. & 0. \\
0. & \frac{1}{4} & 0. & \frac{\varphi}{4} & -\frac{\varphi}{4} & 0. & \frac{1}{4} & 0. \\
0. & \frac{\varphi}{4} & \frac{1}{4} & 0. & 0. & \frac{1}{4} & -\frac{\varphi}{4} & 0. \\
-\frac{\varphi^2}{4} & 0. & 0. & 0. & 0. & 0. & 0. & \frac{1}{4\varphi}
\end{pmatrix}$$

𝕌^2 as the Dot product forms diagonal Cartan-like matrix of this folding matrix 𝕌.𝕌:

```
In[•]:= matCheck@octSym[𝕌.𝕌] // MatrixForm
```

Out[•]//MatrixForm=

$$\begin{pmatrix}
6-2\varphi & 0. & 0. & 0. & 0. & 0. & 0. & \frac{2}{\varphi} \\
0. & 6-2\varphi & 0. & 0. & 0. & 0. & \frac{2}{\varphi} & 0. \\
0. & 0. & 6-2\varphi & 0. & 0. & \frac{2}{\varphi} & 0. & 0. \\
0. & 0. & 0. & 6-2\varphi & \frac{2}{\varphi} & 0. & 0. & 0. \\
0. & 0. & 0. & \frac{2}{\varphi} & 6-2\varphi & 0. & 0. & 0. \\
0. & 0. & \frac{2}{\varphi} & 0. & 0. & 6-2\varphi & 0. & 0. \\
0. & \frac{2}{\varphi} & 0. & 0. & 0. & 0. & 6-2\varphi & 0. \\
\frac{2}{\varphi} & 0. & 0. & 0. & 0. & 0. & 0. & 6-2\varphi
\end{pmatrix}$$



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The original 2012-2014 L/R quadrant preserving as an E_8 to canonical 600-cell (*H_4* family of polytopes) form :

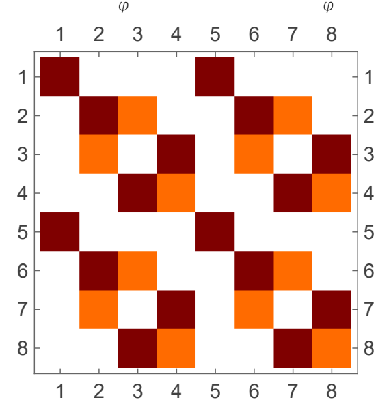
Where L/R quadrants have a Galois transform of $\phi \leftrightarrow -\phi$ with upper left corners exchanging $\Phi \leftrightarrow \phi^2$:

In[•]:= **H4fold =**
$$\begin{pmatrix}
\Phi & 0 & 0 & 0 & \phi^2 & 0 & 0 & 0 \\
0 & \phi & 1 & 0 & 0 & -\phi & 1 & 0 \\
0 & 1 & 0 & \phi & 0 & 1 & 0 & -\phi \\
0 & 0 & \phi & 1 & 0 & 0 & -\phi & 1 \\
\phi^2 & 0 & 0 & 0 & \Phi & 0 & 0 & 0 \\
0 & -\phi & 1 & 0 & 0 & \phi & 1 & 0 \\
0 & 1 & 0 & -\phi & 0 & 1 & 0 & \phi \\
0 & 0 & -\phi & 1 & 0 & 0 & \phi & 1
\end{pmatrix};$$

```
In[•]:= matCheck@H4fold
```

$$\begin{pmatrix}
\varphi & 0 & 0 & 0 & \frac{1}{\varphi^2} & 0 & 0 & 0 \\
0 & \frac{1}{\varphi} & 1 & 0 & 0 & -\frac{1}{\varphi} & 1 & 0 \\
0 & 1 & 0 & \frac{1}{\varphi} & 0 & 1 & 0 & -\frac{1}{\varphi} \\
0 & 0 & \frac{1}{\varphi} & 1 & 0 & 0 & -\frac{1}{\varphi} & 1 \\
\frac{1}{\varphi^2} & 0 & 0 & 0 & \varphi & 0 & 0 & 0 \\
0 & -\frac{1}{\varphi} & 1 & 0 & 0 & \frac{1}{\varphi} & 1 & 0 \\
0 & 1 & 0 & -\frac{1}{\varphi} & 0 & 1 & 0 & \frac{1}{\varphi} \\
0 & 0 & -\frac{1}{\varphi} & 1 & 0 & 0 & \frac{1}{\varphi} & 1
\end{pmatrix}$$

Out[•]=



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```
In[•]:= inN@Det@H4fold
```

Out[•]=

$$\boxed{\begin{matrix}\frac{-64\varphi^3+64\varphi^9}{\varphi^{10}} \\ 37.3499\end{matrix}}$$

```
In[•]:= octSym@Inverse@H4fold // MatrixForm
```

Out[•]//MatrixForm=

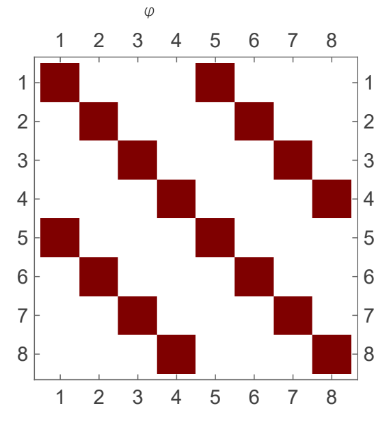
$$\begin{pmatrix}
\frac{\varphi^2}{4} & 0. & 0. & 0. & -\frac{1}{4\varphi} & 0. & 0. & 0. \\
0. & \frac{\varphi}{4} & \frac{1}{4} & 0. & 0. & -\frac{\varphi}{4} & \frac{1}{4} & 0. \\
0. & \frac{1}{4} & 0. & \frac{\varphi}{4} & 0. & \frac{1}{4} & 0. & -\frac{\varphi}{4} \\
0. & 0. & \frac{\varphi}{4} & \frac{1}{4} & 0. & 0. & -\frac{\varphi}{4} & \frac{1}{4} \\
-\frac{1}{4\varphi} & 0. & 0. & 0. & \frac{\varphi^2}{4} & 0. & 0. & 0. \\
0. & -\frac{\varphi}{4} & \frac{1}{4} & 0. & 0. & \frac{\varphi}{4} & \frac{1}{4} & 0. \\
0. & \frac{1}{4} & 0. & -\frac{\varphi}{4} & 0. & \frac{1}{4} & 0. & \frac{\varphi}{4} \\
0. & 0. & -\frac{\varphi}{4} & \frac{1}{4} & 0. & 0. & \frac{\varphi}{4} & \frac{1}{4}
\end{pmatrix}$$

H4fold^2 as the Dot product forms diagonal Cartan-like matrix of this folding matrix H4fold.H4fold:

```
In[•]:= matCheck@octSym[H4fold.H4fold] // MatrixForm
```

Out[•]//MatrixForm=

$$\begin{pmatrix}
6-2\varphi & 0. & 0. & 0. & \frac{2}{\varphi} & 0. & 0. & 0. \\
0. & 6-2\varphi & 0. & 0. & 0. & \frac{2}{\varphi} & 0. & 0. \\
0. & 0. & 6-2\varphi & 0. & 0. & 0. & \frac{2}{\varphi} & 0. \\
0. & 0. & 0. & 6-2\varphi & 0. & 0. & 0. & \frac{2}{\varphi} \\
\frac{2}{\varphi} & 0. & 0. & 0. & 6-2\varphi & 0. & 0. & 0. \\
0. & \frac{2}{\varphi} & 0. & 0. & 0. & 6-2\varphi & 0. & 0. \\
0. & 0. & \frac{2}{\varphi} & 0. & 0. & 0. & 6-2\varphi & 0. \\
0. & 0. & 0. & \frac{2}{\varphi} & 0. & 0. & 0. & 6-2\varphi
\end{pmatrix}$$



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