

Cohl Furey's $\mathbb{R} \otimes \mathbb{C} \otimes \mathbb{H} \otimes \mathbb{O}$ papers:

■ A Computational Analysis by J. Gregory Moxness

Motivation

This is an **updated** analysis of several papers from Cohl Furey related to using complexified real octonions (aka. bi-octonions) in the assignment of Standard Model (SM) Boson and Fermion particles to Clifford algebras and group symmetries. The previous analysis was based on [1-4]:

[1] <https://arxiv.org/abs/1002.1497v5>

[2] <https://arxiv.org/abs/1405.4601v1>

[3] <https://arxiv.org/abs/1603.04078>

[4] <https://arxiv.org/abs/1806.00612>

[5] <https://arxiv.org/abs/1910.08395v1>

This .pdf is a direct output from my Mathematica (MTM) Notebook. I will follow up with a LaTeX paper on the topic soon.

This notebook has code built in to operate symbolically on native MTM reals (\mathbb{R}), complexes ($\mathbb{C}=a+ib$), and quaternionic ($\mathbb{H}=a+ib+cj+dk$) forms, as well as my custom code to handle the octonions ($\mathbb{O}=\{e_0=1, e_1=i, e_2=j, e_3=k, e_4, e_5, e_6, e_7\}$), and now the bi-octonions (which doesn't assign the octonion e_1 to be equivalent to the complex imaginary (i)). That change also applies to the native quaternion assignments where of $e_1 = i, e_2 = j, e_3 = k$) in order to work with quater-octonions. This was a fairly trivial change to make since it simply involves removing the conversion of complex (and quaternion) operators from being involved in the octonionic multiplication (i.e. `SmallCircle[]` or symbolically \circ).

Commenting out that code is shown here:

```
In[2186]:= (* Exclude Complex and Quaternion math to accomodate BiOctonion or Complexified Octonions *)
SmallCircle[a_, 1] := a;
SmallCircle[1, b_] := b;
SmallCircle[a_List, b_List] := octonion@octProduct[a, b];
(*) SmallCircle[a_, b_Quaternion] := a∘quat2oct@b;
SmallCircle[a_Quaternion, b_] := quat2oct@a∘b; (**)
SmallCircle[a_, b_Complex] := a b;
SmallCircle[a_Complex, b_] := a b; (**)
SmallCircle[a_, b_] := If[chkS@{a, b}, a b, oct2List@a∘oct2List@b];
```

Please note that my previous analysis here made the mistake of not commenting out these operations. As such, it was operating on octonions (not complexified bi-octonions), so some of my concerns were resolved based on correcting that error.

This also creates a split in the way conjugation is handled. The MTM native conjugation (`\[Conjugate]` or `*`) handles the imaginary components of \mathbb{R} , \mathbb{C} , and \mathbb{H} (after loading `<<Quaternions``). With the change above, my code operates on octonion symbols (i.e. `oct = {e0, e1, e2, e3, e4, e5, e6, e7}`) as real as required for bi-octonions, but my octonion conjugation (`octCongugate[]` or `x*` (asterisk)) computation operates on octonion symbols as imaginary in addition to the imaginary $\mathbb{C} \in \mathbb{H}$ elements if present (i.e. $i e_n \rightarrow (-i) (-e_n) = i e_n$).

The following code implements the two forms of conjugation introduced in the papers (`*` and `†`):

```

In[2190]= octConjugate@p_ :=
  If[p === s || p === S, Conjugate@p, Total[Conjugate@oct2List@p Join[{1}, -Most@oct]]];
octConjugatePattern : p_* := octConjugate@p;

In[2192]= octConjugate†@p_ :=
  -If[p === s || p === S, Conjugate@p, Total[Conjugate@oct2List@p Join[{1}, -Most@oct]]];
octConjugate†Pattern : p_† := octConjugate†@p;

In[2194]= (* Octonion Conjugation octConjugate (#*) *)
oct

Out[2194]= {e1, e2, e3, e4, e5, e6, e7, e0}

In[2195]= ## & /@ oct

Out[2195]= {-e1, -e2, -e3, -e4, -e5, -e6, -e7, 1}

In[2196]= ##† & /@ oct

Out[2196]= {e1, e2, e3, e4, e5, e6, e7, -1}

```

I've set up this "bi-octonion $\mathbb{R} \otimes \mathbb{C} \otimes \mathbb{H} \otimes \mathbb{O}$ machine" to easily switch from native MTM conjugation and my octonion conjugation.

The exceptions for s & S in this code accommodates their use in these papers as complex elements that only get MTM native treatment, thus avoiding recursive loops when not being reduced with used with the real octonion elements).

The following code implements the s and S in [5]:

```

In[2197]= doConjugate = False;

In[2198]= (* Conjugate *)
sScon = {
  {s*, S},
  {s, S*},
  {s*, S*},
  {s, S}};

In[2199]=

In[2200]= (* octConjugate
  use strings to keep from evaluating conjugate on s & S until needed,
thus avoiding recursive loops *)
sSoct = {
  {"s*", "S"},
  {"s", "S*"},
  {"s*", "S*"},
  {"s", "S"}};

In[2201]= sS := If[doConjugate, sScon, sSoct];

In[2202]= (* Need to avoid infinite recursion in oct2List function
  with complexified octonions which show up with Head=Plus *)
chksS@in_ := MemberQ[DeleteDuplicates@Flatten@sS, in];
chksS@in_List :=
  MemberQ[MemberQ[DeleteDuplicates@Flatten@sS, #] || Head@# === String & /@ in, True];

```

It is also interesting to note that unlike complexes (C), there are actually two forms of quaternion (H) assignments where $k=(ijj \text{ or } jji)$. For octonions, there are 480 possible multiplication tables that are non-trivial (i.e. not simple permutations of the oct symbols). There are 3840 split octonion multiplication tables. These multiplication tables are generated in my code using any of the algorithmically generated 480 possible octonion multiplication tables, as well as visualizing their Fano plane (or Fano cube) mnemonic based on which of the 7 triads selected. This code is also built to accommodate sedenions by Cayley-Dickson doubling, but I digress.

I am very interested here in the suggestion at the very end of Cohl's last paper [5] in the Addendum Section IX(B/C) on Multi-actions splitting spinor spaces, Lie algebras/groups, and Jordan algebras. I suspect having the ability to create a machine (i.e. a symbolic engine such as MTM) to operate on and visualize these structures as hyper-dimensional physical elements is critical to making progress in understanding our Universe more thoroughly.

From [1] Towards a unified theory of ideals (2010)

Select and implement the specific octonion:

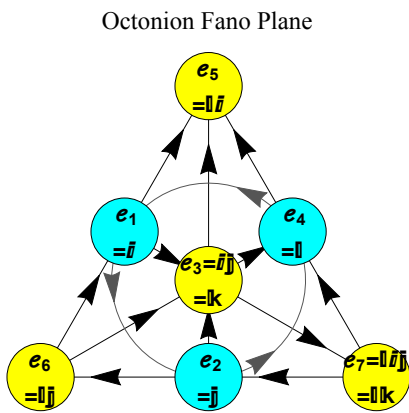
So to set up the analysis, we select the Fano plane used in these papers, which happens to be the same one used by Baez in his work, which is in position 224 "flip→True" in my assignment of the 480 combinations. The algorithm to visualize the Fano plane constructs it differently than the paper, but the triads are correct.

```
In[2204]:= (* Find Cohl Furey's Octonion = Baez's octonion based on triads *)
Position[(setFM[#, True];
StandardForm[triads]) & /@Range@256,
{{1, 2, 4}, {1, 3, 7}, {1, 5, 6}, {2, 3, 5}, {2, 6, 7}, {3, 4, 6}, {4, 5, 7}}]
```

Out[2204]= (224 1)

```
In[2205]:= (* Set the Fano Multiplication matrix to that triad set (224, flip=True)
and display the triads, Fano plane and multiplication table *)
setFM[224, flip = True];
Row@{triads, fanoPlane,
IJKLstyle = False; fmDispe,
IJKLstyle = True;
fmDispe}
```

Out[2206]= $\begin{pmatrix} 1 & 2 & 4 \\ 1 & 3 & 7 \\ 1 & 5 & 6 \\ 2 & 3 & 5 \\ 2 & 6 & 7 \\ 3 & 4 & 6 \\ 4 & 5 & 7 \end{pmatrix}$



$\begin{pmatrix} e_0 \mapsto 1 & e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 \\ e_1 & -1 & e_4 & e_7 & -e_2 & e_6 & -e_5 & -e_3 \\ e_2 & -e_4 & -1 & e_5 & e_1 & -e_3 & e_7 & -e_6 \\ e_3 & -e_7 & -e_5 & -1 & e_6 & e_2 & -e_4 & e_1 \\ e_4 & e_2 & -e_1 & -e_6 & -1 & e_7 & e_3 & -e_5 \\ e_5 & -e_6 & e_3 & -e_2 & -e_7 & -1 & e_1 & e_4 \\ e_6 & e_5 & -e_7 & e_4 & -e_3 & -e_1 & -1 & e_2 \\ e_7 & e_3 & e_6 & -e_1 & e_5 & -e_4 & -e_2 & -1 \end{pmatrix}$	$\begin{pmatrix} e_0 \mapsto 1 & i & j & k & l & li & lj & lk \\ i & -1 & l & lk & -j & lj & -li & -k \\ j & -l & -1 & li & i & -k & lk & -lj \\ k & -lk & -li & -1 & lj & j & -l & i \\ l & j & -i & -lj & -1 & lk & k & -li \\ li & -lj & k & -j & -lk & -1 & i & l \\ lj & li & -lk & l & -k & -i & -1 & j \\ lk & k & lj & -i & li & -l & -j & -1 \end{pmatrix}$
--	--

It should be noted that there can be risk in making assumptions based on operating on a given set of triads→ Fano planes→ multiplication tables. Some of the patterns are specific only to that selected octonion out of the 480 (e.g. A reference in [2] from a reference to the Baez's paper (<https://arxiv.org/abs/math/0105155>) suggesting that index cycling and doubling patterns hold. While interesting, this is only true for that specific octonion multiplication table used in these papers.

Cohl's papers do not reduce the octonionic products (i.e. $e_1 \circ e_2 = e_{12} = e_4$) based on the selected multiplication table outside of using identities common to all imaginary numbers (e.g. $e_n \circ e_m = e_{nm}$, $e_{nm} = -e_{mn}$, and $e_n \circ e_n = -1$).

The following code implements functions needed for basic octonion processing (with some new code for bi-octonions):

```
In[2207]:= (* Octonion math from the triad defined multiplication matrices (fm, fMult, fSign)
using 8D vector representations for x and y, this has the e0=1 first *)
octProduct[a_, b_] := Block[{c3 = Array[0 &, Length@fm]},
  Do[c3[[If[i ≠ j, fMult[[i, j]] + 1, 1]]] += fSign[[i, j]] a[[i]] b[[j]],
    {i, Length@fm}, {j, Length@fm}];
  Chop@c3] /. s1Rep;
```

```
In[2208]:= (**)
Unprotect@Conjugate;
eConjugatePattern : (e_n_)^* := (* -**) e_n;
Protect@Conjugate;
```

```
In[2211]:= Unprotect@Power;
eExponentPattern : e_n_-^m := (-1)^(m-1) e_n;
Protect@Power;
```

Beginning the introduction of octonion chains including octonion reduction using the selected triad multiplication table.

```
In[2214]:= (* Some simple octProduct math *)
```

```
In[2215]:= e1 ° e2
```

```
Out[2215]:= e4
```

```
In[2216]:= e2 ° e3
```

```
Out[2216]:= e5
```

```
In[2217]:= e1 ° (e2 ° e3)
```

```
Out[2217]:= e6
```

```
In[2218]:= e3 ° (e2 ° e1)
```

```
Out[2218]:= -e6
```

```
In[2219]:= e6 + i ° e2
```

```
Out[2219]:= e4 + e6
```

Implementing Left vs. Right chains and precedence mapping function (f):

I constructed a function f to build the left (L) parenthetical multiplication precedence, which can be reversed by simply providing a "right→True" parameter to perform the right (R) operations. In addition, it takes a "right→both" parameter to both reverse the order and the parenthetical precedence. Note the use of the SmallCircle "°" for doing proper octonion multiplication. While the paper does not show the evaluation of the octonionic multiplication, I do that in order to verify the logic.

```
In[2220]:= (* Define the parenthetical reverse nesting with Octonion multiplication table operation ° *)
Clear@f;
f[in_List, right_ : False] := Module[{inFlat = Flatten@in, out = 1},
  Do[out = If[right === both, inFlat[[1]] ° out,
    If[right, out ° inFlat[[1]],
      Reverse[inFlat] [[1]] ° out]], {1, Length@inFlat}];
  Simplify@out];
```

Using an example from [2] Generations: Three Prints, in Colour (2014) paragraph "Octonionic Chains", we evaluate this function first w/o a defined octonion product table to see the proper parenthetical precedence is

obtained for left and right operations:

```
In[2222]= Clear[SmallCircle]
```

```
In[2223]= f@{e3, e4, e6 + i e2}
```

```
Out[2223]= e3 ° (e4 ° ((e6 + i ° e2) ° 1))
```

Redefine SmallCircle and re-evaluate:

```
In[2224]= SmallCircle[a_, 1] := a;
SmallCircle[1, b_] := b;
SmallCircle[a_List, b_List] := octonion@octProduct[a, b];
SmallCircle[a_, b_] := If[ChksS@{a, b}, a b, oct2List@a ° oct2List@b];
```

```
In[2228]= f@{e3, e4, e6 + i e2}
(* or in Mathematica postfix notation as a right to left functor as used in the paper *)
{e3, e4, e6 + i e2} // f
```

```
Out[2228]= -1 + i e7
```

```
Out[2229]= -1 + i e7
```

The associator function:

```
In[2230]= associator[a_List, b_List, c_List] := (a ° b) ° c - a ° (b ° c);
associator[a_, b_, c_] := associator[oct2List@a, oct2List@b, oct2List@c];
```

If the associator is zero, it doesn't show the non-associative property.

```
In[2232]= associator[e3, e4, e5]
```

```
Out[2232]= -2 e1
```

```
In[2233]= associator[e3, e4, e6]
```

```
Out[2233]= 0
```

From [2]: Generations: Three Prints, in Colour (2014)

The following implements / checks the ν identities:

```
In[2234]= ν = - 1/2 f[{e3, e4, e6 + i e2}, True]
```

```
Out[2234]= 1/2 (1 + i e7)
```

```
In[2235]= (* Normal Conjugation - only negates i without changing e_n *)
ν*
```

```
Out[2235]= 1/2 (1 - i e7)
```

```
In[2236]= (* Cross Conjugation -
negates the real compared to normal (or both compared to octConjugation *)
ν†
```

```
Out[2236]= -1/2 - i e7/2
```

```
In[2237]= (* octConjugation - negates both i and e_n resulting in no change on i e_n *)
ν*
```

```
Out[2237]= 1/2 + i e7/2
```

In[2238]:= **(* No Conjugation Identities *)**

Simplify $\left[\frac{1}{2} (1 + \mathbf{i} e_7) == \nu == \nu \circ \nu\right]$

Out[2238]= True

In[2239]:= **(* octConjugation Identities *)**

Simplify $\left[\frac{1}{2} (1 + \mathbf{i} e_7) == \nu^* == \nu^* \circ \nu^*\right]$

Out[2239]= True

In[2240]:= **(* Normal Conjugation Identities *)**

Simplify $\left[\frac{1}{2} (1 - \mathbf{i} e_7) == \nu^* == \nu^* \circ \nu^*\right]$

$\theta == \nu \circ \nu^* == \nu^* \circ \nu$

Out[2240]= True

Out[2241]= True

In[2242]:= **(* Cross Conjugation Identities *)**

In[2243]:= ν^\dagger

Out[2243]= $-\frac{1}{2} - \frac{i e_7}{2}$

In[2244]:= $\nu^\dagger \circ \nu^\dagger$

Out[2244]= $\frac{1}{2} + \frac{i e_7}{2}$

In[2245]:= **(* Cross Conjugation Identities *)**

$\frac{1}{2} + \frac{\mathbf{i} e_7}{2} == -\nu^\dagger == \nu^\dagger \circ \nu^\dagger$

$\nu^\dagger == \nu \circ \nu^\dagger == \nu^\dagger \circ \nu$

Out[2245]= True

Out[2246]= True

In[2247]:= $\nu \odot \nu^\dagger$

Out[2247]= 0

From [3]: Charge quantization from a number operator (2016)

The following implements / checks the new basis $\{\alpha_1, \alpha_2, \alpha_3\}$:

In[2248]:= **Clear** $[\alpha, \alpha_1, \alpha_2, \alpha_3]$

In[2249]:= $\alpha_1 = \frac{1}{2} (-e_5 + \mathbf{i} e_4)$

Out[2249]= $\frac{1}{2} (-e_5 + i e_4)$

In[2250]:= **(* Normal Conjugation *)**

α_1^*

Out[2250]= $\frac{1}{2} (-e_5 - i e_4)$

In[2251]:= (*** octConjugation ***)
 $\alpha 1^*$

$$\text{Out[2251]} = \frac{e_5}{2} + \frac{i e_4}{2}$$

In[2252]:= (*** Cross Conjugation ***)
 $\alpha 1^\dagger$

$$\text{Out[2252]} = -\frac{e_5}{2} - \frac{i e_4}{2}$$

In[2253]:= **$\alpha 2 = \frac{1}{2} (-e_3 + \mathbf{i} e_1)$**

$$\text{Out[2253]} = \frac{1}{2} (-e_3 + i e_1)$$

In[2254]:= (*** Normal Conjugation ***)
 $\alpha 2^*$

$$\text{Out[2254]} = \frac{1}{2} (-e_3 - i e_1)$$

In[2255]:= **$\alpha 2^*$**

$$\text{Out[2255]} = \frac{e_3}{2} + \frac{i e_1}{2}$$

In[2256]:= **$\alpha 2^\dagger$**

$$\text{Out[2256]} = -\frac{e_3}{2} - \frac{i e_1}{2}$$

In[2257]:= **$\alpha 3 = \frac{1}{2} (-e_6 + \mathbf{i} e_2)$**

$$\text{Out[2257]} = \frac{1}{2} (-e_6 + i e_2)$$

In[2258]:= (*** Normal Conjugation ***)
 $\alpha 3^*$

$$\text{Out[2258]} = \frac{1}{2} (-e_6 - i e_2)$$

In[2259]:= **$\alpha 3^*$**

$$\text{Out[2259]} = \frac{e_6}{2} + \frac{i e_2}{2}$$

In[2260]:= **$\alpha 3^\dagger$**

$$\text{Out[2260]} = -\frac{e_6}{2} - \frac{i e_2}{2}$$

In[2261]:= **Simplify[f@{ $\alpha 1$, $\alpha 2$ } == - $\mathbf{i} \alpha 3^\dagger$]**
Simplify[f@{ $\alpha 3$, $\alpha 1$ } == - $\mathbf{i} \alpha 2^\dagger$]
Simplify[f@{ $\alpha 2$, $\alpha 3$ } == - $\mathbf{i} \alpha 1^\dagger$]

Out[2261]= True

Out[2262]= True

Out[2263]= True

Applying f and octonion multiplication reduces the following to zero:

In[2264]= **Simplify**@{-{ α_2^\dagger , α_1 }, -{ α_1^\dagger , α_2 }}
 {-f@{ α_2^\dagger , α_1 }, -f@{ α_1^\dagger , α_2 }}

Out[2264]=
$$\begin{pmatrix} \frac{1}{2}(i e_1 + e_3) & \frac{1}{2}(e_5 - i e_4) \\ \frac{1}{2}(i e_4 + e_5) & \frac{1}{2}(e_3 - i e_1) \end{pmatrix}$$

Out[2265]= {0, 0}

In[2266]= **{#[[1]] \circ #[[2]]} & /@ {{ α_1 , α_2 }, { α_2 , α_3 }, { α_1 , α_3 }}**
{#[[1]] † \circ #[[2]] † } & /@ {{ α_1 , α_2 }, { α_2 , α_3 }, { α_1 , α_3 }}
{#[[1]], #[[2]] † , #[[1]] \circ #[[2]] † , #[[2]] \circ #[[1]] † } & /@ {{ α_1 , α_2 }, { α_2 , α_3 }, { α_1 , α_3 }}

Out[2266]=
$$\begin{pmatrix} \frac{i e_6}{2} - \frac{e_7}{2} \\ \frac{i e_5}{2} - \frac{e_4}{2} \\ \frac{e_1}{2} - \frac{i e_3}{2} \end{pmatrix}$$

Out[2267]=
$$\begin{pmatrix} -\frac{e_2}{2} - \frac{i e_6}{2} \\ -\frac{e_4}{2} - \frac{i e_5}{2} \\ \frac{e_1}{2} + \frac{i e_3}{2} \end{pmatrix}$$

Out[2268]=
$$\begin{pmatrix} \frac{1}{2}(i e_4 - e_5) & -\frac{i e_1}{2} - \frac{e_3}{2} & 0 & 0 \\ \frac{1}{2}(i e_1 - e_3) & -\frac{i e_5}{2} - \frac{e_6}{2} & 0 & 0 \\ \frac{1}{2}(i e_4 - e_5) & -\frac{i e_2}{2} - \frac{e_6}{2} & 0 & 0 \end{pmatrix}$$

The following implements / checks ω identities:

In[2269]= **(* ω (or is it ν) is a negative charge particle that converts up \rightarrow
 down via right multiplication (i.e. W^-) *)**

$\omega = f@{\alpha_1, \alpha_2, \alpha_3}$

Out[2269]= $\frac{1}{2}(-e_7 + i)$

In[2270]= **(* octConjugation *)**
 ω^*

Out[2270]= $\frac{e_7}{2} - \frac{i}{2}$

In[2271]= **(* Normal Conjugation *)**
 ω^*

Out[2271]= $\frac{1}{2}(-e_7 - i)$

In[2272]= ω^\dagger

Out[2272]= $-\frac{e_7}{2} + \frac{i}{2}$

In[2273]= **(* ω^\dagger (or is it ν^*) is a positive charge particle that converts down \rightarrow
 up via right multiplication (i.e. W^+) *)**

Expand[$\omega == \omega^\dagger$]

Out[2273]= True

In[2274]= **f@{ α_1^\dagger , α_2^\dagger , α_3^\dagger }**

Out[2274]= $\frac{1}{2}(-e_7 - i)$

In[2275]= **$\nu = \text{Simplify}[-f@{\omega, \omega^\dagger}]$**

Out[2275]= $\frac{1}{2}(1 + i e_7)$

In[2276]= **Expand**[-v == f@{ω, ω†} == f@{ω†, ω} == v†]

Out[2276]= True

In[2277]= **To**String[Subscript["2α", #], TraditionalForm] <>
 "=" <> **To**String[Expand[2 α#], TraditionalForm], Spacer@10,
ToString[Subsuperscript["2α", #, "†"], TraditionalForm] "=" <>
ToString[Subsuperscript["2α", #, "*"], TraditionalForm] <> "=", Expand[2 α#†], Spacer@10,
ToString[Subsuperscript["2α", #, "oct*"], TraditionalForm] <>
 "=" <> **To**String[Expand[2 α#*], TraditionalForm] } & /@ Range@3

Out[2277]=
$$\begin{pmatrix} 2\alpha_1=2\alpha_1 & 2\alpha_1^\dagger=2\alpha_1^\dagger & 2e_1 & 2\alpha_1^{\text{Oct}^*}=-2e_1 \\ 2\alpha_2=2\alpha_2 & 2\alpha_2^\dagger=2\alpha_2^\dagger & 2e_2 & 2\alpha_2^{\text{Oct}^*}=-2e_2 \\ 2\alpha_3=2\alpha_3 & 2\alpha_3^\dagger=2\alpha_3^\dagger & 2e_3 & 2\alpha_3^{\text{Oct}^*}=-2e_3 \end{pmatrix}$$

In[2278]= ω† ◦ α#† & /@ Range@3

Out[2278]=
$$\left\{ -\frac{e_3}{2} + \frac{i e_1}{2}, -\frac{e_6}{2} + \frac{i e_2}{2}, \frac{e_1}{2} + \frac{i e_3}{2} \right\}$$

In[2279]= α# ◦ v & /@ Range@3

Out[2279]=
$$\left\{ \frac{e_1}{2} - \frac{i e_3}{2}, \frac{e_2}{2} - \frac{i e_6}{2}, \frac{e_3}{2} + \frac{i e_1}{2} \right\}$$

The following implements / checks the new basis $\Lambda_1 - \Lambda_8$:

In[2280]= **Λ1** = -f@{α2†, α1} - f@{α1†, α2}

Out[2280]= 0

In[2281]= **Simplify**[{-{α2†, α1}, -{α1†, α2}}]
 {-f@{α2†, α1}, -f@{α1†, α2}}

Out[2281]=
$$\begin{pmatrix} \frac{1}{2}(i e_1 + e_3) & \frac{1}{2}(e_5 - i e_4) \\ \frac{1}{2}(i e_4 + e_5) & \frac{1}{2}(e_3 - i e_1) \end{pmatrix}$$

Out[2282]= {0, 0}

In[2283]= **Λ2** = i f@{α2†, α1} - i f@{α1†, α2}

Out[2283]= 0

In[2284]= **Expand**@{{α2†, α2}, -{α1†, α1}}
Expand@{f@{α2†, α2}, -f@{α1†, α1}}

Out[2284]=
$$\begin{pmatrix} -\frac{i e_1}{2} - \frac{e_3}{2} & \frac{i e_1}{2} - \frac{e_3}{2} \\ \frac{i e_4}{2} + \frac{e_5}{2} & \frac{e_5}{2} - \frac{i e_4}{2} \end{pmatrix}$$

Out[2285]=
$$\left\{ -\frac{1}{2} + \frac{i e_7}{2}, \frac{1}{2} - \frac{i e_7}{2} \right\}$$

In[2286]= **Λ3** = f@{α2†, α2} - f@{α1†, α1}

Out[2286]= 0

In[2287]= **Expand**@{-{α1†, α3}, -{α3†, α1}}
Expand@{-f@{α1†, α3}, -f@{α3†, α1}}

Out[2287]=
$$\begin{pmatrix} \frac{i e_4}{2} + \frac{e_5}{2} & \frac{e_6}{2} - \frac{i e_7}{2} \\ \frac{i e_7}{2} + \frac{e_6}{2} & \frac{e_5}{2} - \frac{i e_4}{2} \end{pmatrix}$$

Out[2288]= {0, 0}

In[2289]= **Λ4** = -f@{α1†, α3} - f@{α3†, α1}

Out[2289]= 0

In[2290]= **Expand@**{-i { $\alpha 1^\dagger$, $\alpha 3$ }, i { $\alpha 3^\dagger$, $\alpha 1$ }}
Expand@{-f@{ $\alpha 1^\dagger$, $\alpha 3$ }, f@{ $\alpha 3^\dagger$, $\alpha 1$ }}

$$\text{Out[2290]= } \begin{pmatrix} \frac{i e_5}{2} - \frac{e_4}{2} & \frac{e_2}{2} + \frac{i e_6}{2} \\ \frac{e_2}{2} - \frac{i e_6}{2} & -\frac{e_4}{2} - \frac{i e_5}{2} \end{pmatrix}$$

Out[2291]= {0, 0}

In[2292]= $\Delta 5 = -i f@{\alpha 1^\dagger, \alpha 3} + i f@{\alpha 3^\dagger, \alpha 1}$

Out[2292]= 0

In[2293]= **Expand@**{- { $\alpha 3^\dagger$, $\alpha 2$ }, - { $\alpha 2^\dagger$, $\alpha 3$ }}
Expand@{-f@{ $\alpha 3^\dagger$, $\alpha 2$ }, -f@{ $\alpha 2^\dagger$, $\alpha 3$ }}

$$\text{Out[2293]= } \begin{pmatrix} \frac{i e_2}{2} + \frac{e_6}{2} & \frac{e_3}{2} - \frac{i e_1}{2} \\ \frac{i e_1}{2} + \frac{e_3}{2} & \frac{e_6}{2} - \frac{i e_2}{2} \end{pmatrix}$$

Out[2294]= {0, 0}

In[2295]= $\Delta 6 = -f@{\alpha 3^\dagger, \alpha 2} - f@{\alpha 2^\dagger, \alpha 3}$

Out[2295]= 0

In[2296]= **Expand@**{ { $\alpha 3^\dagger$, $\alpha 2$ }, - { $\alpha 2^\dagger$, $\alpha 3$ }}
Expand@{f@{ $\alpha 3^\dagger$, $\alpha 2$ }, -f@{ $\alpha 2^\dagger$, $\alpha 3$ }}

$$\text{Out[2296]= } \begin{pmatrix} -\frac{i e_2}{2} - \frac{e_6}{2} & \frac{i e_1}{2} - \frac{e_3}{2} \\ \frac{i e_1}{2} + \frac{e_3}{2} & \frac{e_6}{2} - \frac{i e_2}{2} \end{pmatrix}$$

Out[2297]= {0, 0}

In[2298]= $\Delta 7 = i f@{\alpha 3^\dagger, \alpha 2} - i f@{\alpha 2^\dagger, \alpha 3}$

Out[2298]= 0

In[2299]= **Expand@**{ { $\alpha 1^\dagger$, $\alpha 1$ }, { $\alpha 2^\dagger$, $\alpha 2$ }, -2 { $\alpha 3^\dagger$, $\alpha 3$ }}
Expand@{f@{ $\alpha 1^\dagger$, $\alpha 1$ }, f@{ $\alpha 2^\dagger$, $\alpha 2$ }, -2 f@{ $\alpha 3^\dagger$, $\alpha 3$ }}

$$\text{Out[2299]= } \begin{pmatrix} -\frac{i e_4}{2} - \frac{e_5}{2} & \frac{i e_4}{2} - \frac{e_5}{2} \\ -\frac{i e_1}{2} - \frac{e_3}{2} & \frac{i e_1}{2} - \frac{e_3}{2} \\ i e_2 + e_6 & e_6 - i e_2 \end{pmatrix}$$

$$\text{Out[2300]= } \left\{ -\frac{1}{2} + \frac{i e_7}{2}, -\frac{1}{2} + \frac{i e_7}{2}, 1 - i e_7 \right\}$$

In[2301]= $\Delta 8 = -\frac{1}{\sqrt{3}} (f@{\alpha 1^\dagger, \alpha 1} + f@{\alpha 2^\dagger, \alpha 2} - 2 f@{\alpha 3^\dagger, \alpha 3})$

Out[2301]= 0

Commutation ($x \odot y$) and AntiCommutation ($x \ominus y$)

In[2302]= **commutator**[x_, y_] := **Simplify@Chop**[x \odot y - y \odot x] /. **slRep**

If[! **doClifford**,

CircleDot[x_, y_] := **commutator**[x, y];

derivation[x_, y_][a_] := (x \odot y) \odot a - 3 **associator**[x, y, a],

derivation[x_, y_][a_] := **commutator**[**commutator**[x, y], a] - 3 **associator**[x, y, a]] /. **slRep**;

In[2304]= **antiCommutator**[x_, y_] := **Simplify@Chop**[x \odot y + y \odot x] /. **slRep**;

CircleMinus[x_, y_] := **antiCommutator**[x, y];

Commutation checks

$$\text{In}[2306]:= \sum_{j=2}^6 \sum_{i=1}^{i-1} \mathbf{f@oct}[\{i, j\}] \circ \mathbf{f@oct}[\{j, i\}]$$

Out[2306]= 0

$$\text{In}[2307]:= \sum_{j=2}^6 \sum_{i=1}^{i-1} (\mathbf{f@oct}[\{i, j\}] + \mathbf{f@oct}[\{j, i\}])$$

Out[2307]= 0

AntiCommutation checks

$$\text{In}[2308]:= \alpha_1 \ominus \alpha_1$$

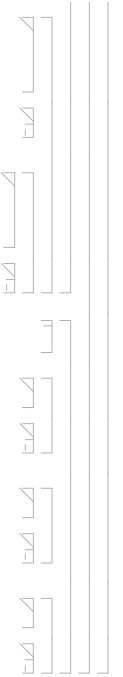
Out[2308]= 0

$$\text{In}[2309]:= \alpha_1 \ominus \alpha_2^\dagger$$

Out[2309]= 0

$$\text{In}[2310]:= \alpha_1 \ominus \alpha_1^\dagger$$

Out[2310]= -1



From [4]: $(SU(3))_C \times (SU(2))_L \times (U(1))_Y \times (U(1))_X$ as a symmetry of division algebraic ladder operators (2018)

4.3 Quarks and leptons as minimal left ideals (A one generation SM)

\mathcal{V} & \mathcal{E} using $+/-$, $1/i$, e_0/e_7 and $Q = 0$ using no α_n

$$\text{In[2311]: } \mathcal{V} = \text{Expand@}\mathcal{v}$$

$$\text{Out[2311]: } \frac{1}{2} + \frac{i e_7}{2}$$

$$\text{In[2312]: } \overline{\mathcal{V}} = \text{Expand@}\mathcal{v}^*$$

$$\text{Out[2312]: } \frac{1}{2} - \frac{i e_7}{2}$$

$$\text{In[2313]: } \mathcal{E} = \text{Expand@f@}\{\alpha_1, \alpha_2, \alpha_3, \mathcal{v}^*\}$$

$$\text{Out[2313]: } \frac{e_6}{2} - \frac{i e_2}{2}$$

$$\text{In[2314]: } \overline{\mathcal{E}} = \text{Expand@f@}\{\alpha_1^\dagger, \alpha_2^\dagger, \alpha_3^\dagger, \mathcal{v}\}$$

$$\text{Out[2314]: } \frac{e_6}{2} + \frac{i e_2}{2}$$

$\mathcal{U}r$ & $\mathcal{D}r$ using $+/-$, $1/i$, e_4/e_5 and $Q = \mp 1/3$ or $\pm 2/3$ using 1 or 2 α_n (respectively)

$$\text{In[2315]: } \mathcal{U}r = -\text{Expand@f@}\{\alpha_3^\dagger, \alpha_2^\dagger, \mathcal{v}\}$$

$$\text{Out[2315]: } \frac{e_5}{2} - \frac{i e_4}{2}$$

$$\text{In[2316]: } \overline{\mathcal{U}r} = \text{Expand@f@}\{\alpha_3, \alpha_2, \mathcal{v}^*\}$$

$$\text{Out[2316]: } -\frac{e_5}{2} - \frac{i e_4}{2}$$

$$\text{In[2317]: } \mathcal{D}r = -\text{Expand@f@}\{\alpha_1, \mathcal{v}^*\}$$

$$\text{Out[2317]: } -\frac{e_1}{2} - \frac{i e_3}{2}$$

$$\text{In[2318]: } \overline{\mathcal{D}r} = \text{Expand@f@}\{\alpha_1^\dagger, \mathcal{v}\}$$

$$\text{Out[2318]: } \frac{e_1}{2} - \frac{i e_3}{2}$$

$\mathcal{U}g$ & $\mathcal{D}g$ using $+/-$, $1/i$, e_1/e_3 and $Q = \mp 1/3$ or $\pm 2/3$ using 1 or 2 α_n (respectively)

$$\text{In[2319]: } \mathcal{U}g = -\text{Expand@f@}\{\alpha_1^\dagger, \alpha_3^\dagger, \mathcal{v}\}$$

$$\text{Out[2319]: } -\frac{e_7}{2} + \frac{i}{2}$$

$$\text{In[2320]: } \overline{\mathcal{U}g} = \text{Expand@f@}\{\alpha_1, \alpha_3, \mathcal{v}^*\}$$

$$\text{Out[2320]: } \frac{e_7}{2} + \frac{i}{2}$$

$$\text{In[2321]: } \mathcal{D}g = -\text{Expand@f@}\{\alpha_2, \mathcal{v}^*\}$$

$$\text{Out[2321]: } -\frac{e_2}{2} - \frac{i e_6}{2}$$

$$\text{In[2322]: } \overline{\mathcal{D}g} = \text{Expand@f@}\{\alpha_2^\dagger, \mathcal{v}\}$$

$$\text{Out[2322]: } \frac{e_2}{2} - \frac{i e_6}{2}$$

$\mathcal{U}b$ & $\mathcal{D}b$ using $+/-$, $1/i$, e_2/e_6 and $Q = \mp 1/3$ or $\pm 2/3$ using 1 or 2 α_n (respectively)

In[2323]= $\mathcal{U}b = -\text{Expand@f@}\{\alpha_1^\dagger, \alpha_2^\dagger, v\}$

Out[2323]= $-\frac{e_4}{2} - \frac{i e_5}{2}$

In[2324]= $\overline{\mathcal{U}b} = \text{Expand@f@}\{\alpha_1, \alpha_2, v^*\}$

Out[2324]= $\frac{e_4}{2} - \frac{i e_5}{2}$

In[2325]= $\mathcal{D}b = -\text{Expand@f@}\{\alpha_3, v^*\}$

Out[2325]= $-\frac{e_3}{2} + \frac{i e_1}{2}$

In[2326]= $\overline{\mathcal{D}b} = \text{Expand@f@}\{\alpha_3^\dagger, v\}$

Out[2326]= $\frac{e_3}{2} + \frac{i e_1}{2}$

From [5] : Three generations, two unbroken gauge symmetries, and one eight-dimensional algebra (2019)

4.2 Clifford algebraic structure (Mf)

In[2327]= (* This is a switch used to either include the i in 1-V, 3-V, & 5-V or not *)
addI = True;

In[2328]= (* 4.2 Clifford algebraic structure Mf *)

In[2329]= (* 1 element of 0-V - $e_0 = 1$ *)
f@{oct[[8]]}

Out[2329]= e_0

In[2330]= (* 6 elements of 1-V - $e_1+e_2+e_3+e_4+e_5+e_6$ (*) *)

$$\sum_{i=1}^6 \text{If}[\text{addI}, i, e_0] \circ \text{f@}\{\text{oct}[[i]]\}$$

(* The number of unique dimensional elements after octonion multiplication *)
Length@%

Out[2330]= $i e_1 + i e_2 + i e_3 + i e_4 + i e_5 + i e_6$

Out[2331]= 6

In[2332]= (* 15 elements of 2-V -

$$e_1 \circ e_2 + e_1 \circ e_3 + e_1 \circ e_4 + e_1 \circ e_5 + e_1 \circ e_6 + e_2 \circ e_3 + e_2 \circ e_4 + e_2 \circ e_5 + e_2 \circ e_6 + e_3 \circ e_4 + e_3 \circ e_5 + e_3 \circ e_6 + e_4 \circ e_5 + e_4 \circ e_6 + e_5 \circ e_6 \text{ *)}$$

$$\sum_{j=2}^6 \sum_{i=1}^{j-1} \text{f@oct}[[\{i, j\}]]$$

Length@%

(* The total of the coefficients = the number of elements in the original calculation minus the dimensions that cancel each other *)

Total@Abs@oct2List@%%

Out[2332]= $2 e_1 + 2 e_6 + 3 e_7$

Out[2333]= 3

Out[2334]= 7

```
In[2335]= (* 20 elements of 3-V - e1°(e2°e3)+e1°(e2°e4)+e1°(e2°e5)+e1°(e2°e6)+e1°(e3°e4)+
e1°(e3°e5)+e1°(e3°e6)+e1°(e4°e5)+e1°(e4°e6)+e1°(e5°e6)+e2°(e3°e4)+e2°(e3°e5)+e2°(e3°e6)+
e2°(e4°e5)+e2°(e4°e6)+e2°(e5°e6)+e3°(e4°e5)+e3°(e4°e6)+e3°(e5°e6)+e4°(e5°e6) *)

$$\sum_{k=3}^6 \sum_{j=2}^{k-1} \sum_{i=1}^{j-1} \text{If}[\text{addI}, i, e_0] \circ f @ \text{oct}[\{i, j, k\}]$$

```

Length@%

Total@Abs@oct2List@%%

Out[2335]= $2ie_2 - 2ie_3 - 4i$

Out[2336]= 3

Out[2337]= 8

```
In[2338]= (* 1 element of 6-V - e7 = e1°(e2°(e3°(e4°(e5°e6)))) *)
f@oct[Range@6]
```

Out[2338]= e_7

```
In[2339]= (* Missing 4-V and 5-V for full Mf *)
{%, %%%, Out@-7, Out@-9, Out@-10}
Length@Total@%
Total@Abs@oct2List@Total@%%
```

Out[2339]= $\{e_7, 2ie_2 - 2ie_3 - 4i, 2e_1 + 2e_6 + 3e_7, ie_1 + ie_2 + ie_3 + ie_4 + ie_5 + ie_6, e_0\}$

Out[2340]= 9

Out[2341]= $10 + 2\sqrt{5} + \sqrt{17}$

```
In[2342]= (* Clifford Mf reduced *)
```

```
In[2343]= (* 8 element combining the 2 single 0-V and 6-V with the 6 element 1-V,
adding the factor of i as in the paper if addI is True and it is in 1-V,
3-V, or 5-V) *)
```

$$\sum_{i=1}^8 \text{If}[\text{addI} \&\& i < 7, i, e_0] \circ f @ \{\text{oct}[i]\}$$

Length@%

Out[2343]= $ie_1 + ie_2 + ie_3 + ie_4 + ie_5 + ie_6 + e_7 + 1$

Out[2344]= 8

```
In[2345]= (* 21 elements combining 15 contained in 2-V and 6 contained in 5-
V - e1°e2+e1°e3+e1°e4+e1°e5+e1°e6+e1°e7+e2°e3+e2°e4+e2°e5+e2°e6+
e2°e7+e3°e4+e3°e5+e3°e6+e3°e7+e4°e5+e4°e6+e4°e7+e5°e6+e5°e7+e6°e7 *)
```

$$\sum_{j=2}^7 \sum_{i=1}^{j-1} \text{If}[\text{addI} \&\& j == 7, i, e_0] \circ f @ \text{oct}[\{i, j\}]$$

Length@%

Total@Abs@oct2List@%%

Out[2345]= $(2+i)e_1 + ie_2 - ie_3 + ie_4 - ie_5 + (2-i)e_6 + 3e_7$

Out[2346]= 7

Out[2347]= $7 + 2\sqrt{5}$

```

In[2348]:= (* 35 elements - combining the 20 element 3-V and 15 element 4-V -
  e1°(e2°e3)+e1°(e2°e4)+e1°(e2°e5)+e1°(e2°e6)+e1°(e2°e7)+e1°(e3°e4)+e1°(e3°e5)+
  e1°(e3°e6)+e1°(e3°e7)+e1°(e4°e5)+e1°(e4°e6)+e1°(e4°e7)+e1°(e5°e6)+e1°(e5°e7)+
  e1°(e6°e7)+e2°(e3°e4)+e2°(e3°e5)+e2°(e3°e6)+e2°(e3°e7)+e2°(e4°e5)+e2°(e4°e6)+
  e2°(e4°e7)+e2°(e5°e6)+e2°(e5°e7)+e2°(e6°e7)+e3°(e4°e5)+e3°(e4°e6)+e3°(e4°e7)+
  e3°(e5°e6)+e3°(e5°e7)+e3°(e6°e7)+e4°(e5°e6)+e4°(e5°e7)+e4°(e6°e7)+e5°(e6°e7) *)
  Sum[Sum[Sum[If[addI && k < 7, i, e0] ° f@oct[{i, j, k}]]
    {k=3, j=2, i=1}]]
Length@%
Total@Abs@oct2List@%%

Out[2348]= (-2 + 2 i) e2 + (2 - 2 i) e3 - (3 + 4 i)

Out[2349]= 3

Out[2350]= 5 + 4√2

In[2351]:= (* Sum of the 0V to 6-V *)
%% + Out@-6 + Out@-8
Length@%
Total@Abs@oct2List@%%

Out[2351]= (2 + 2 i) e1 - (2 - 4 i) e2 + (2 - 2 i) e3 + 2 i e4 + 2 e6 + 4 e7 - (2 + 4 i)

Out[2352]= 7

Out[2353]= 8 + 4√2 + 4√5

In[2354]:= (* Not the same due to the missing 4-V and 5-V in the Mf of the first section *)
%
Out@-14
% == %%

Out[2354]= 8 + 4√2 + 4√5

Out[2355]= 10 + 2√5 + √17

Out[2356]= False

In[2357]:= (* Adding 4-V {-2 e2+2 e3-3} and 5-V {e1+e2-e3+e4-e5-e6} from below
  to first section confirms equality with this section's Mf reduced *)
0 == Out@-6 - Total[Join[Out@-18, (* 4-V *) {-2 e2, 2 e3, -3},
  (* 5-V *) If[addI, i, e0] ° # & /@ {e1, e2, -e3, e4, -e5, -e6}]]] /. e0 → 1

Out[2357]= True

The following reconfirms the previous 2 section's 0-V to 6-V combinations:

In[2358]:= (* 0-V *)

In[2359]:= f@{oct[8]} /. e0 → 1
Abs@%

Out[2359]= 1

Out[2360]= 1

In[2361]:= (* 1-V = 6 element *)

```

In[2362]= $\sum_{i=1}^6 \text{If}[\text{addI}, \mathbf{i}, e_0] \circ f @ \{\text{oct}[\mathbf{i}]\}$
Length@%

Out[2362]= $i e_1 + i e_2 + i e_3 + i e_4 + i e_5 + i e_6$

Out[2363]= 6

In[2364]= (* 2-V = 15 element -
 $e_1 \circ e_2 + e_1 \circ e_3 + e_1 \circ e_4 + e_1 \circ e_5 + e_1 \circ e_6 + e_2 \circ e_3 + e_2 \circ e_4 + e_2 \circ e_5 + e_2 \circ e_6 + e_3 \circ e_4 + e_3 \circ e_5 + e_3 \circ e_6 + e_4 \circ e_5 + e_4 \circ e_6 + e_5 \circ e_6$ *)

In[2365]= $\sum_{j=2}^6 \sum_{i=1}^{j-1} f @ \text{oct}[\{\mathbf{i}, \mathbf{j}\}]$
Length@%
Total@Abs@oct2List@%%

Out[2365]= $2 e_1 + 2 e_6 + 3 e_7$

Out[2366]= 3

Out[2367]= 7

In[2368]= $f @ \text{oct}[\{\#\{1\}, \#\{2\}\}] \& /@ \text{Subsets}[\text{Range}@6, \{2\}]$
Length@%
Total@%%
Total@Abs@oct2List@%

Out[2368]= $\{e_4, e_7, -e_2, e_6, -e_5, e_5, e_1, -e_3, e_7, e_6, e_2, -e_4, e_7, e_3, e_1\}$

Out[2369]= 15

Out[2370]= $2 e_1 + 2 e_6 + 3 e_7$

Out[2371]= 7

In[2372]= (* 3-V = 20 element (without the factor of \mathbf{i} as in the paper) -
 $e_1 \circ (e_2 \circ e_3) + e_1 \circ (e_2 \circ e_4) + e_1 \circ (e_2 \circ e_5) + e_1 \circ (e_2 \circ e_6) + e_1 \circ (e_3 \circ e_4) + e_1 \circ (e_3 \circ e_5) +$
 $e_1 \circ (e_3 \circ e_6) + e_1 \circ (e_4 \circ e_5) + e_1 \circ (e_4 \circ e_6) + e_1 \circ (e_5 \circ e_6) + e_2 \circ (e_3 \circ e_4) + e_2 \circ (e_3 \circ e_5) + e_2 \circ (e_3 \circ e_6) +$
 $e_2 \circ (e_4 \circ e_5) + e_2 \circ (e_4 \circ e_6) + e_2 \circ (e_5 \circ e_6) + e_3 \circ (e_4 \circ e_5) + e_3 \circ (e_4 \circ e_6) + e_3 \circ (e_5 \circ e_6) + e_4 \circ (e_5 \circ e_6)$
 (with the factor of \mathbf{i} as in the paper) -
 $\mathbf{i} e_1 \circ (e_2 \circ e_3) + \mathbf{i} e_1 \circ (e_2 \circ e_4) + \mathbf{i} e_1 \circ (e_2 \circ e_5) + \mathbf{i} e_1 \circ (e_2 \circ e_6) + \mathbf{i} e_1 \circ (e_3 \circ e_4) + \mathbf{i} e_1 \circ (e_3 \circ e_5) + \mathbf{i} e_1 \circ (e_3 \circ e_6) +$
 $\mathbf{i} e_1 \circ (e_4 \circ e_5) + \mathbf{i} e_1 \circ (e_4 \circ e_6) + \mathbf{i} e_1 \circ (e_5 \circ e_6) + \mathbf{i} e_2 \circ (e_3 \circ e_4) + \mathbf{i} e_2 \circ (e_3 \circ e_5) + \mathbf{i} e_2 \circ (e_3 \circ e_6) + \mathbf{i} e_2 \circ (e_4 \circ e_5) +$
 $\mathbf{i} e_2 \circ (e_4 \circ e_6) + \mathbf{i} e_2 \circ (e_5 \circ e_6) + \mathbf{i} e_3 \circ (e_4 \circ e_5) + \mathbf{i} e_3 \circ (e_4 \circ e_6) + \mathbf{i} e_3 \circ (e_5 \circ e_6) + \mathbf{i} e_4 \circ (e_5 \circ e_6)$ *)

In[2373]= $\sum_{k=3}^6 \sum_{j=2}^{k-1} \sum_{i=1}^{j-1} \text{If}[\text{addI}, \mathbf{i}, e_0] \circ f @ \text{oct}[\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}]$
Length@%
Total@Abs@oct2List@%%

Out[2373]= $2 i e_2 - 2 i e_3 - 4 i$

Out[2374]= 3

Out[2375]= 8


```
In[2376]= If[addI, i, e0] ° f@oct[#{#1, #2, #3}] & /@ Subsets[Range@6, {3}]
Length@%
Total@%%
Total@Abs@oct2List@%
```

```
Out[2376]= {i e6, -i, -i e7, -i e3, -i e5, i e4, i e2, -i e3, i e7, -i, i e7, -i, -i e1, -i e6, i e5, -i e4, i e1, -i, -i e7, i e2}
```

```
Out[2377]= 20
```

```
Out[2378]= 2 i e2 - 2 i e3 - 4 i
```

```
Out[2379]= 8
```

```
In[2380]= (* 4-V = 15 element {e1 ° (e2 ° e7), e1 ° (e3 ° e7), e1 ° (e4 ° e7), e1 ° (e5 ° e7), e1 ° (e6 ° e7), e2 ° (e3 ° e7), e2 ° (e4 ° e7),
e2 ° (e5 ° e7), e2 ° (e6 ° e7), e3 ° (e4 ° e7), e3 ° (e5 ° e7), e3 ° (e6 ° e7), e4 ° (e5 ° e7), e4 ° (e6 ° e7), e5 ° (e6 ° e7)} *)
```

```
In[2381]= f@oct[#{#1, #2, 7}] & /@ Subsets[Range@6, {2}]
Length@%
Total@%%
Total@Abs@oct2List@%
```

```
Out[2381]= {e5, -1, -e6, -e2, e4, -e4, e3, e1, -1, -e2, e6, -e5, -1, -e1, e3}
```

```
Out[2382]= 15
```

```
Out[2383]= -2 e2 + 2 e3 - 3
```

```
Out[2384]= 7
```

```
In[2385]= (* 5-V = 6 element (without the factor of i as in the paper) -
{e1 ° e7, e2 ° e7, e3 ° e7, e4 ° e7, e5 ° e7, e6 ° e7} *)
```

```
In[2386]= If[addI, i, e0] ° f@oct[#{#, 7}] & /@ Range@6
Length@%
Total@%%
Total@Abs@oct2List@%
```

```
Out[2386]= {-i e3, -i e6, i e1, -i e5, i e4, i e2}
```

```
Out[2387]= 6
```

```
Out[2388]= i e1 + i e2 - i e3 + i e4 - i e5 - i e6
```

```
Out[2389]= 6
```

```
In[2390]= (* 6-V = e1 ° (e2 ° (e3 ° (e4 ° (e5 ° e6)))) *)
```

```
In[2391]= f@oct[Range@6]
Total@Abs@oct2List@%
```

```
Out[2391]= e7
```

```
Out[2392]= 1
```

```

In[2393]=

(* Summary of the 0-V to 6-V elements in the original calculations *)
{%%, Out@-7, Out@-12, Out@-17, Out@-25, Out@-31, Out@-33} // ColumnForm
Total@Flatten@First@%

Out[2393]= e7
{-i e3, -i e6, i e1, -i e5, i e4, i e2}
{e5, -1, -e6, -e2, e4, -e4, e3, e1, -1, -e2, e6, -e5, -1, -e1, e3}
{i e6, -i, -i e7, -i e3, -i e5, i e4, i e2, -i e3, i e7, -i, i e7, -i, -i e1, -i e6, i e5, -i e4, i e1, -i, -i e7, i e2}
{e4, e7, -e2, e6, -e5, e5, e1, -e3, e7, e6, e2, -e4, e7, e3, e1}
i e1 + i e2 + i e3 + i e4 + i e5 + i e6
1

Out[2394]= (2 + 2 i) e1 - (2 - 4 i) e2 + (2 - 2 i) e3 + 2 i e4 + 2 e6 + 4 e7 - (2 + 4 i)

In[2395]= (* The 0-V to 6-
V totals and grand total number of elements in the original calculations *)
{%%%, Out@-8, Out@-13, Out@-18, Out@-26, Out@-32, Out@-35}
Total@%

Out[2395]= {1, 6, 15, 20, 15, 6, 1}

Out[2396]= 64

In[2397]= (* The 0-V to 6-V totals and grand total of the coefficients = the number of elements
in the original calculations minus the dimensions that cancel each other *)
{Out@-5, Out@-8, Out@-13, Out@-18, Out@-26, Out@-34, Out@-37}
Total@%

Out[2397]= {1, 6, 7, 8, 7, 6, 1}

Out[2398]= 36

In[2399]= (* The 0-V to 6-V sum of all level elements combined
with that combination's total and grand total coefficients *)
{Out@-8, Out@-11, Out@-16, Out@-21, Out@-29, Out@-37, Out@-39} // ColumnForm
Total@Flatten@First@%

Out[2399]= e7
i e1 + i e2 - i e3 + i e4 - i e5 - i e6
-2 e2 + 2 e3 - 3
2 i e2 - 2 i e3 - 4 i
2 e1 + 2 e6 + 3 e7
i e1 + i e2 + i e3 + i e4 + i e5 + i e6
1

Out[2400]= (2 + 2 i) e1 - (2 - 4 i) e2 + (2 - 2 i) e3 + 2 i e4 + 2 e6 + 4 e7 - (2 + 4 i)

In[2401]= (* As it should be, the Mf reduced and sum of 0-V to 6-
V is the same (with or without the factor of i as in the paper) *)
% - Out@-7

Out[2401]= 0

In[2402]= (* There are a total of 20 coefficients after octonion multiplication and combining,
which is a delta from the 64 basis elements of 44 *)
Total@Abs@oct2List@%%

Out[2402]= 8 + 4  $\sqrt{2}$  + 4  $\sqrt{5}$ 

```

In[2403]= (* Without combining the 0-V to 7-V rows together,
 there is a delta of 28 coefficients after octonion multiplication and combining,
 (or 26 if building a factor of i into f@ on all Octonion elements)
 *)
 Out@-7 - Out@-5

Out[2403]= 28

The following visualizes the 0-V to 6-V derived from MTM subsets:

In[2404]= gather = GatherBy[Subsets@Range@6, Length@# &]
 Length /@ %

Out[2404]= $\left\{ \left\{ \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{pmatrix} \right\}, \begin{pmatrix} 1 & 2 \\ 1 & 3 \\ 1 & 4 \\ 1 & 5 \\ 1 & 6 \\ 2 & 3 \\ 2 & 4 \\ 2 & 5 \\ 2 & 6 \\ 3 & 4 \\ 3 & 5 \\ 3 & 6 \\ 4 & 5 \\ 4 & 6 \\ 5 & 6 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 5 \\ 1 & 2 & 3 & 6 \\ 1 & 2 & 4 & 5 \\ 1 & 2 & 4 & 6 \\ 1 & 2 & 5 & 6 \\ 1 & 3 & 4 & 5 \\ 1 & 3 & 4 & 6 \\ 1 & 3 & 5 & 6 \\ 1 & 4 & 5 & 6 \\ 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 6 \\ 2 & 3 & 5 & 6 \\ 2 & 4 & 5 & 6 \\ 3 & 4 & 5 & 6 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 6 \\ 1 & 2 & 3 & 5 & 6 \\ 1 & 2 & 4 & 5 & 6 \\ 1 & 3 & 4 & 5 & 6 \\ 2 & 3 & 4 & 5 & 6 \end{pmatrix}, (1 \ 2 \ 3 \ 4 \ 5 \ 6) \right\}$

Out[2405]= {1, 6, 15, 20, 15, 6, 1}

In[2406]= invGather = (Complement[Range@7, #] & /@ #) & /@ gather

Out[2406]= $\left\{ (1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7), \begin{pmatrix} 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 3 & 4 & 5 & 6 & 7 \\ 1 & 2 & 4 & 5 & 6 & 7 \\ 1 & 2 & 3 & 5 & 6 & 7 \\ 1 & 2 & 3 & 4 & 6 & 7 \\ 1 & 2 & 3 & 4 & 5 & 7 \end{pmatrix}, \begin{pmatrix} 3 & 4 & 5 & 6 & 7 \\ 2 & 4 & 5 & 6 & 7 \\ 2 & 3 & 5 & 6 & 7 \\ 2 & 3 & 4 & 6 & 7 \\ 2 & 3 & 4 & 5 & 7 \\ 1 & 4 & 5 & 6 & 7 \\ 1 & 3 & 5 & 6 & 7 \\ 1 & 3 & 4 & 6 & 7 \\ 1 & 3 & 4 & 5 & 7 \\ 1 & 2 & 5 & 6 & 7 \\ 1 & 2 & 4 & 6 & 7 \\ 1 & 2 & 4 & 5 & 7 \\ 1 & 2 & 3 & 6 & 7 \\ 1 & 2 & 3 & 5 & 7 \\ 1 & 2 & 3 & 4 & 7 \end{pmatrix}, \begin{pmatrix} 4 & 5 & 6 & 7 \\ 3 & 5 & 6 & 7 \\ 3 & 4 & 6 & 7 \\ 3 & 4 & 5 & 7 \\ 2 & 5 & 6 & 7 \\ 2 & 4 & 6 & 7 \\ 2 & 4 & 5 & 7 \\ 2 & 3 & 6 & 7 \\ 2 & 3 & 5 & 7 \\ 2 & 3 & 4 & 7 \\ 1 & 5 & 6 & 7 \\ 1 & 4 & 6 & 7 \\ 1 & 4 & 5 & 7 \\ 1 & 3 & 6 & 7 \\ 1 & 3 & 5 & 7 \\ 1 & 3 & 4 & 7 \\ 1 & 2 & 6 & 7 \\ 1 & 2 & 5 & 7 \\ 1 & 2 & 4 & 7 \\ 1 & 2 & 3 & 7 \end{pmatrix}, \begin{pmatrix} 5 & 6 & 7 \\ 4 & 6 & 7 \\ 4 & 5 & 7 \\ 3 & 6 & 7 \\ 3 & 5 & 7 \\ 3 & 4 & 7 \\ 2 & 6 & 7 \\ 2 & 5 & 7 \\ 2 & 4 & 7 \\ 2 & 3 & 7 \\ 1 & 6 & 7 \\ 1 & 5 & 7 \\ 1 & 4 & 7 \\ 1 & 3 & 7 \\ 1 & 2 & 7 \end{pmatrix}, \begin{pmatrix} 6 & 7 \\ 5 & 7 \\ 4 & 7 \\ 3 & 7 \\ 2 & 7 \\ 1 & 7 \end{pmatrix}, (7) \right\}$

```
In[2407]= (* Now show these as reduced bi-octonion elements L/R→Bottom/Top *)
addI = (*)False**)True(**);
Table[{
  If[addI && EvenQ@i, i, e0] ◦ f@oct[[gather[[i, j]]],
  If[addI && EvenQ@i, i, e0] ◦ f@oct[[invGather[[i, j]]], {i, 7}, {j, Length@gather[[i]]} /. e0 → 1
Reverse[Total@# & /@%]
(*)
  Total@#&/@%/ColumnForm
  Total@%/ColumnForm**)
```

$$\text{Out[2408]= } \left\{ (1 \ -1), \begin{pmatrix} i e_1 & i e_1 \\ i e_2 & -i e_2 \\ i e_3 & i e_3 \\ i e_4 & -i e_4 \\ i e_5 & i e_5 \\ i e_6 & -i e_6 \end{pmatrix}, \begin{pmatrix} e_4 & e_4 \\ e_7 & -e_7 \\ -e_2 & -e_2 \\ e_6 & -e_6 \\ -e_5 & -e_5 \\ e_5 & e_5 \\ e_1 & -e_1 \\ -e_3 & -e_3 \\ e_7 & -e_7 \\ e_6 & e_6 \\ e_2 & -e_2 \\ -e_4 & -e_4 \\ e_7 & e_7 \\ e_3 & -e_3 \\ e_1 & e_1 \end{pmatrix}, \begin{pmatrix} i e_6 & -i e_6 \\ -i & -i \\ -i e_7 & i e_7 \\ -i e_3 & -i e_3 \\ -i e_5 & i e_5 \\ i e_4 & i e_4 \\ i e_2 & -i e_2 \\ -i e_3 & i e_3 \\ i e_7 & i e_7 \\ -i & i \\ i e_7 & i e_7 \\ -i & i \\ -i e_1 & -i e_1 \\ -i e_6 & -i e_6 \\ i e_5 & -i e_5 \\ -i e_4 & -i e_4 \\ i e_1 & -i e_1 \\ -i & -i \\ -i e_7 & i e_7 \\ i e_2 & i e_2 \end{pmatrix}, \begin{pmatrix} -e_3 & e_3 \\ -e_1 & -e_1 \\ 1 & -1 \\ e_5 & -e_5 \\ e_6 & e_6 \\ e_2 & -e_2 \\ -1 & -1 \\ -e_1 & e_1 \\ e_3 & e_3 \\ e_4 & -e_4 \\ -e_4 & e_4 \\ -e_2 & -e_2 \\ e_6 & -e_6 \\ -1 & -1 \\ -e_5 & e_5 \end{pmatrix}, \begin{pmatrix} i e_2 & i e_2 \\ -i e_4 & i e_4 \\ -i e_5 & -i e_5 \\ -i e_1 & i e_1 \\ -i e_6 & -i e_6 \\ i e_3 & -i e_3 \end{pmatrix}, (e_7 \ e_7) \right\}$$

$$\text{Out[2409]= } \begin{pmatrix} e_7 & e_7 \\ -i e_1 + i e_2 + i e_3 - i e_4 - i e_5 - i e_6 & i e_1 + i e_2 - i e_3 + i e_4 - i e_5 - i e_6 \\ -2 e_1 + 2 e_6 - 1 & -2 e_2 + 2 e_3 - 3 \\ 2 i e_2 - 2 i e_3 - 4 i & -2 i e_1 - 2 i e_6 + 4 i e_7 \\ 2 e_1 + 2 e_6 + 3 e_7 & -2 e_2 - 2 e_3 - e_7 \\ i e_1 + i e_2 + i e_3 + i e_4 + i e_5 + i e_6 & i e_1 - i e_2 + i e_3 - i e_4 + i e_5 - i e_6 \\ 1 & -1 \end{pmatrix}$$

Notice that the symmetry of the structure is more complicated than before applying the octonion Product.

Associativity of Composition of Maps

```
In[2410]= Clear[F, G, H]
```

We clear f and define it as fff to allow for optional evaluations using replace (/ . f -> fff)

```
In[2411]= Clear@f
```

```
(* Define the parenthetical reverse nesting
with Octonion multiplication table operation ◦ *)
fff[in_List, right_:False] := Module[{inFlat = Flatten@in, out = 1},
  Do[out = If[right === both, inFlat[[l]] ◦ out,
  If[right, out ◦ inFlat[[l]],
  Reverse[inFlat][[l]] ◦ out]], {l, Length@inFlat}];
Simplify@out];
```

```
In[2413]= F := f@{e2, e3, #} /. f → fff &;
G := f@{e3, #} /. f → fff &;
H := f@{e4, e6, e7, #} /. f → fff &;
```

```
In[2416]:= {F@#, G@#, H@#} & /@ {e0}
f[Flatten@%] /. f -> fff
f[Flatten@%, True] /. f -> fff
f[Flatten@%%, both] /. f -> fff
```

```
Out[2416]= (e5 e3 -e1)
```

```
Out[2417]= e4
```

```
Out[2418]= -e4
```

```
Out[2419]= -e4
```

```
In[2420]:= (* Left to Right *)
(F@e0) // G // H
```

```
Out[2420]= -e4
```

```
In[2421]:= (* Right to Left *)
F@(G@(H@e0))
```

```
Out[2421]= -e4
```

Redefine Λ & g as functions: Λ_n for $n = 1 - 8$ g_n for $n = 9 - 14$ (note: similar structure with $\frac{i}{2}$ factor)

```
In[2422]:= Clear[ $\Lambda$ ,  $\Lambda_1$ ,  $\Lambda_2$ ,  $\Lambda_3$ ,  $\Lambda_4$ ,  $\Lambda_5$ ,  $\Lambda_6$ ,  $\Lambda_7$ ,  $\Lambda_8$ ,  $g$ ]
```

```
In[2423]:=  $\Lambda$ [n_, ff_] := (**)Total[(**) $f$ @{#, ff} & /@  $\Lambda_n$ (**)] (**);
 $g$ [n_, ff_] := (**)Total[(**) $f$ @{ $g_n$ , ff} & /@  $g_n$ (**)] (**);
```

```
In[2425]:= (*  $\Lambda_1 \rightarrow -e_6/2$  *)
```

```
In[2426]:=  $\Lambda_1 = \frac{i}{2} \{\{e_1, e_5\}, -\{e_3, e_4\}\};$ 
 $\Lambda_1$ @ff_ :=  $\Lambda$ [1, ff];
 $\Lambda_1$ @e0 /. f -> fff
```

```
Out[2428]=  $-\frac{e_6}{2}$ 
```

```
In[2429]:= (*  $\Lambda_1 f \rightarrow v$  *)
```

```
In[2430]:=  $\Lambda$ [1, v] /. f -> fff
```

```
Out[2430]=  $\frac{1}{4} i (e_2 + i e_6)$ 
```

```
In[2431]:= (*  $\Lambda_2 \rightarrow \pm e_2/2 \rightarrow 0$  *)
```

```
In[2432]:=  $\Lambda_2 = \frac{i}{2} \{-\{e_1, e_4\}, -\{e_3, e_5\}\};$ 
 $\Lambda_2$ @ff_ :=  $\Lambda$ [2, ff];
 $\Lambda_2$ @e0 /. f -> fff
```

```
Out[2434]= 0
```

```
In[2435]:= (*  $\Lambda_2 f \rightarrow v$  *)
```

```
In[2436]:=  $\Lambda$ [2, v] /. f -> fff
```

```
Out[2436]=  $\frac{1}{8} (-e_2 - i e_6) + \frac{1}{8} (e_2 + i e_6)$ 
```

```
In[2437]:= (*  $\Lambda_3 \rightarrow -e_7/2$  *)
```

In[2438]:= $\Lambda_3 = \frac{i}{2} \{-e_4, e_5\}, \{e_1, e_3\}$;
 $\Lambda_3@ff_ := \Lambda[3, ff]$;
 $\Lambda_3@e_0 /. f \rightarrow fff$

Out[2440]= $-\frac{e_7}{2}$

In[2441]:= $(* \Lambda_3 f \rightarrow v *)$

In[2442]:= $\Lambda[3, v] /. f \rightarrow fff$

Out[2442]= $\frac{1}{4}(-e_7 + i)$

In[2443]:= $(* \Lambda_4 \rightarrow \pm e_3 / 2 \rightarrow 0 *)$

In[2444]:= $\Lambda_4 = \frac{i}{2} \{e_2, e_5\}, \{e_4, e_6\}$;
 $\Lambda_4@ff_ := \Lambda[4, ff]$;
 $\Lambda_4@e_0 /. f \rightarrow fff$

Out[2446]= 0

In[2447]:= $(* \Lambda_4 f \rightarrow v *)$

In[2448]:= $\Lambda[4, v] /. f \rightarrow fff$

Out[2448]= $\frac{1}{8}i(e_1 + ie_3) + \frac{1}{8}(e_3 - ie_1)$

In[2449]:= $(* \Lambda_5 \rightarrow -e_1 / 2 *)$

In[2450]:= $\Lambda_5 = \frac{i}{2} \{-e_5, e_6\}, \{e_2, e_4\}$;
 $\Lambda_5@ff_ := \Lambda[5, ff]$;
 $\Lambda_5@e_0 /. f \rightarrow fff$

Out[2452]= $-\frac{e_1}{2}$

In[2453]:= $(* \Lambda_4 f \rightarrow v *)$

In[2454]:= $\Lambda[5, v] /. f \rightarrow fff$

Out[2454]= $\frac{1}{4}(-e_1 - ie_3)$

In[2455]:= $(* \Lambda_6 \rightarrow \pm e_5 / 2 \rightarrow 0 *)$

In[2456]:= $\Lambda_6 = \frac{i}{2} \{e_1, e_6\}, \{e_2, e_3\}$;
 $\Lambda_6@ff_ := \Lambda[6, ff]$;
 $\Lambda_6@e_0 /. f \rightarrow fff$

Out[2458]= 0

In[2459]:= $(* \Lambda_6 f \rightarrow v *)$

In[2460]:= $\Lambda[6, v] /. f \rightarrow fff$

Out[2460]= $\frac{1}{8}i(e_4 + ie_5) + \frac{1}{8}(e_5 - ie_4)$

In[2461]:= $(* \Lambda_7 \rightarrow \mp e_4 / 2 \rightarrow 0 *)$

$$\text{In[2462]}:= \Lambda_7 = \frac{i}{2} \{ \{e_1, e_2\}, \{e_3, e_6\} \};$$

$$\Lambda_7@ff_ := \Lambda[7, ff];$$

$$\Lambda_7@e_0 /. f \rightarrow fff$$

$$\text{Out[2464]}= 0$$

$$\text{In[2465]}:= (* \Lambda_7 f \rightarrow v *)$$

$$\text{In[2466]}:= \Lambda[7, v] /. f \rightarrow fff$$

$$\text{Out[2466]}= \frac{1}{8} (-e_4 - i e_5) + \frac{1}{8} (e_4 + i e_5)$$

$$\text{In[2467]}:= (* \Lambda_8 \rightarrow -e_7/2 *)$$

$$\text{In[2468]}:= \Lambda_8 = \frac{i}{2\sqrt{3}} \{ \{e_1, e_3\}, \{e_4, e_5\}, -2 \{e_2, e_6\} \};$$

$$\Lambda_8@ff_ := \Lambda[8, ff];$$

$$\Lambda_8@e_0 /. f \rightarrow fff$$

$$\text{Out[2470]}= -\frac{e_7}{2}$$

$$\text{In[2471]}:= (* \Lambda_8 f \rightarrow v *)$$

$$\text{In[2472]}:= \Lambda[8, v] /. f \rightarrow fff$$

$$\text{Out[2472]}= \frac{1}{4} (-e_7 + i)$$

$$\text{In[2473]}:= (* g_9 \rightarrow -e_6/12^2 *)$$

$$\text{In[2474]}:= g_9 = -\frac{i}{2\sqrt{3}} \{ \{e_1, e_5\}, \{e_3, e_4\}, 2 \{e_2, e_7\} \};$$

$$g_9@ff_ := g[9, ff];$$

$$g_9@e_0 /. f \rightarrow fff$$

$$\text{Out[2476]}= -\frac{e_6}{144}$$

$$\text{In[2477]}:= (* g_9 f \rightarrow v *)$$

$$\text{In[2478]}:= g[9, v] /. f \rightarrow fff$$

$$\text{Out[2478]}= -\frac{1}{288} i (e_2 - i e_6)$$

$$\text{In[2479]}:= (* g_{10} \rightarrow +e_2/12^2 *)$$

$$\text{In[2480]}:= g_{10} = \frac{i}{2\sqrt{3}} \{ -\{e_4, e_1\}, -\{e_3, e_5\}, 2 \{e_6, e_7\} \};$$

$$g_{10}@ff_ := g[10, ff];$$

$$g_{10}@e_0 /. f \rightarrow fff$$

$$\text{Out[2482]}= \frac{e_2}{144}$$

$$\text{In[2483]}:= (* g_{10} f \rightarrow v *)$$

$$\text{In[2484]}:= g[10, v] /. f \rightarrow fff$$

$$\text{Out[2484]}= \frac{1}{288} (e_2 - i e_6)$$

$$\text{In[2485]}:= (* g_{11} \rightarrow +e_3/12^2 *)$$

$$\text{In[2486]: } \mathbf{g}_{11} = \frac{i}{2\sqrt{3}} \{-\{e_4, e_6\}, -\{e_5, e_2\}, 2\{e_7, e_1\}\};$$

$\mathbf{g}_{11}@\mathbf{ff}_ := \mathbf{g}[11, \mathbf{ff}];$

$\mathbf{g}_{11}@e_0 /. \mathbf{f} \rightarrow \mathbf{fff}$

$$\text{Out[2488]: } \frac{e_3}{144}$$

$\text{In[2489]: } (* \mathbf{g}_{11}\mathbf{f} \rightarrow \mathbf{v} *)$

$\text{In[2490]: } \mathbf{g}[11, \mathbf{v}] /. \mathbf{f} \rightarrow \mathbf{fff}$

$$\text{Out[2490]: } \frac{1}{288} i (e_1 - i e_3)$$

$\text{In[2491]: } (* \mathbf{g}_{12} \rightarrow +e_1 / 12^2 *)$

$$\text{In[2492]: } \mathbf{g}_{12} = \frac{i}{2\sqrt{3}} \{-\{e_2, e_4\}, -\{e_5, e_6\}, 2\{e_3, e_7\}\};$$

$\mathbf{g}_{12}@\mathbf{ff}_ := \mathbf{g}[12, \mathbf{ff}];$

$\mathbf{g}_{12}@e_0 /. \mathbf{f} \rightarrow \mathbf{fff}$

$$\text{Out[2494]: } \frac{e_1}{144}$$

$\text{In[2495]: } (* \mathbf{g}_{12}\mathbf{f} \rightarrow \mathbf{v} *)$

$\text{In[2496]: } \mathbf{g}[12, \mathbf{v}] /. \mathbf{f} \rightarrow \mathbf{fff}$

$$\text{Out[2496]: } \frac{1}{288} (e_1 - i e_3)$$

$\text{In[2497]: } (* \mathbf{g}_{13} \rightarrow +e_5 / 24 *)$

$$\text{In[2498]: } \mathbf{g}_{13} = \frac{i}{2\sqrt{3}} \{-\{e_6, e_1\}, -\mathbf{f}@\{e_2, e_3\}, 2\mathbf{f}@\{e_7, e_4\}\};$$

$\mathbf{g}_{13}@\mathbf{ff}_ := \mathbf{g}[13, \mathbf{ff}];$

$\mathbf{g}_{13}@e_0 /. \mathbf{f} \rightarrow \mathbf{fff}$

$$\text{Out[2500]: } \frac{e_5}{24}$$

$\text{In[2501]: } (* \mathbf{g}_{13}\mathbf{f} \rightarrow \mathbf{v} *)$

$\text{In[2502]: } \mathbf{g}[13, \mathbf{v}] /. \mathbf{f} \rightarrow \mathbf{fff}$

$$\text{Out[2502]: } -\frac{1}{48} i (e_4 + i e_5)$$

$\text{In[2503]: } (* \mathbf{g}_{14} \rightarrow +e_4 / 12^2 *)$

$$\text{In[2504]: } \mathbf{g}_{14} = \frac{i}{2\sqrt{3}} \{-\{e_1, e_2\}, -\{e_6, e_3\}, 2\{e_5, e_7\}\};$$

$\mathbf{g}_{14}@\mathbf{ff}_ := \mathbf{g}[14, \mathbf{ff}];$

$\mathbf{g}_{14}@e_0 /. \mathbf{f} \rightarrow \mathbf{fff}$

$$\text{Out[2506]: } \frac{e_4}{144}$$

$\text{In[2507]: } (* \mathbf{g}_{14}\mathbf{f} \rightarrow \mathbf{v} *)$

$\text{In[2508]: } \mathbf{g}[14, \mathbf{v}] /. \mathbf{f} \rightarrow \mathbf{fff}$

$$\text{Out[2508]: } \frac{1}{288} (e_4 - i e_5)$$

Three generation particle models

Redefine raising & lowering operators $\{\alpha_1, \alpha_2, \alpha_3\}$ as functions of $n = 1 - 3$ and f

In[2509]:= **Clear@ α ;**

In[2510]:= **$\alpha[n_, ff_] := \alpha_n \circ ff$;**

In[2511]:= **$\alpha_1 = \alpha 1$**

Out[2511]= $\frac{1}{2}(-e_5 + i e_4)$

In[2512]:= **$\alpha_2 = \alpha 2$**

Out[2512]= $\frac{1}{2}(-e_3 + i e_1)$

In[2513]:= **$\alpha_3 = \alpha 3$**

Out[2513]= $\frac{1}{2}(-e_6 + i e_2)$

In[2514]:= **$\alpha[1, e_0] /. f \rightarrow fff$**

Out[2514]= $-\frac{e_5}{2} + \frac{i e_4}{2}$

In[2515]:= **$\alpha[1, e_0]^\dagger /. f \rightarrow fff$**

Out[2515]= $-\frac{e_5}{2} - \frac{i e_4}{2}$

In[2516]:= **$\alpha[1, v] /. f \rightarrow fff$**

Out[2516]= 0

In[2517]:= **$\alpha 1 \ominus \alpha 2^\dagger /. f \rightarrow fff$**

Out[2517]= 0

In[2518]:= **$\alpha_1 \ominus \alpha_1^\dagger /. f \rightarrow fff$**

Out[2518]= -1

In[2519]:= **$\alpha 1 \ominus \alpha 1^\dagger /. f \rightarrow fff$**

Out[2519]= -1

In[2520]:= **$\alpha[1, e_0] \ominus \alpha[1, e_0]^\dagger /. f \rightarrow fff$**

Out[2520]= -1

Define SU(1) Q as a function of α_n $n = 1 - 3$ and f

In[2521]:= **Clear@su1Q;**

In[2522]:= **$su1Q := \sum_{i=1}^3 \frac{1}{3} f @ \{\alpha_i^\dagger, \alpha_i, \#\} \&;$**

```
In[2523]= su1Q@e0 / . f → fff
%†
%%*
```

```
Out[2523]=  $\frac{1}{2} i (e_7 + i)$ 
```

```
Out[2524]=  $\frac{1}{2} - \frac{i e_7}{2}$ 
```

```
Out[2525]=  $-\frac{1}{2} + \frac{i e_7}{2}$ 
```

```
In[2526]= su1Q@e4
Simplify@%
```

```
Out[2526]=  $\frac{1}{3} f\left(\left\{-\frac{e_3}{2} - \frac{i e_1}{2}, \frac{1}{2}(-e_3 + i e_1), e_4\right\}\right) + \frac{1}{3} f\left(\left\{-\frac{e_5}{2} - \frac{i e_4}{2}, \frac{1}{2}(-e_5 + i e_4), e_4\right\}\right) + \frac{1}{3} f\left(\left\{-\frac{e_6}{2} - \frac{i e_2}{2}, \frac{1}{2}(-e_6 + i e_2), e_4\right\}\right)$ 
```

```
Out[2527]=  $\frac{1}{3} \left( f\left(\left\{-\frac{1}{2} i (e_1 - i e_3), \frac{1}{2} i (e_1 + i e_3), e_4\right\}\right) + f\left(\left\{-\frac{1}{2} i (e_4 - i e_5), \frac{1}{2} i (e_4 + i e_5), e_4\right\}\right) + f\left(\left\{-\frac{1}{2} i (e_2 - i e_6), \frac{1}{2} i (e_2 + i e_6), e_4\right\}\right) \right)$ 
```

```
In[2528]= f@{α#†, α#} / . f → fff & /@Range@3
```

```
Out[2528]=  $\left\{\frac{1}{2} i (e_7 + i), \frac{1}{2} i (e_7 + i), \frac{1}{2} i (e_7 + i)\right\}$ 
```

Define exp G as a function of $j = 1 - 8 \rightarrow r_j$ and f

```
In[2529]= Clear@G
```

```
In[2530]= r0 = 2.;
```

```
r1 = r2 = r3 = r4 = r5 = r6 = r7 = r8 = 1.;
G[j_, ff_] := ei rj Δ[j, ff] + i r0 su1Q@ff;
```

```
In[2533]= i r1 Δ[1, 1] / . f → fff
i r0 su1Q@e0 / . f → fff
```

```
Out[2533]=  $(0. - 0.5 i) e_6$ 
```

```
Out[2534]=  $(-1. + 0. i) (e_7 + i)$ 
```

```
In[2535]= G[1, 1] / . f → fff
```

```
Out[2535]=  $\exp((0. - 0.5 i) e_6 - (1. + 0. i) (e_7 + i))$ 
```

```
In[2536]= Chop@{#, G[1, 1] α# G[1, 1]-1, G[1, 1] α#† G[1, 1]-1} & /@Range@3 / . f → fff
```

```
Out[2536]=  $\begin{pmatrix} 1 & 0.5 (i e_4 - e_5) & 1. \left(-\frac{i e_4}{2} - \frac{e_5}{2}\right) \\ 2 & 0.5 (i e_1 - e_3) & 1. \left(-\frac{i e_1}{2} - \frac{e_3}{2}\right) \\ 3 & 0.5 (i e_2 - e_6) & 1. \left(-\frac{i e_2}{2} - \frac{e_6}{2}\right) \end{pmatrix}$ 
```

Define the 3 generations of quarks and leptons (2014 and 2019 versions)

```
In[2537]= Clear[u1, u2, u3, Δ1Δ2, Δ4Δ5, Δ6Δ7,
u1r, u1g, u1b, u2r, u2g, u2b, u3r, u3g, u3b,
d123rgb, d1r, d1g, d1b, d2r, d2g, d2b, d3r, d3g, d3b,
v1, v2, v3, e123, e1, e2, e3]
```

```
In[2538]= u1@oct_ := {sS[[2], Flatten@{Total@oct}, sS[[4]]};
u2@oct_ := {sS[[3], Flatten@{Total@oct}, sS[[1]]};
u3@oct_ := {sS[[4], Flatten@{Total@oct}, sS[[1]]};
```

```
In[2541]= Δnm[nm_List] := f@# & /@ Flatten[{nm}, 1];
```

```
In[2542]= (* u1r construction / deconstruction *)
```

In[2543]:= $-\mathbf{i} \Lambda_6 \{-1, 1\}$ // StandardForm

$\Delta_{nm}[-\mathbf{i} \Lambda_6] \{-1, 1\}$

Total@% /. f → fff

Out[2543]//StandardForm=

$$\left\{ \left\{ -\frac{e_1}{2}, -\frac{e_6}{2} \right\}, \left\{ \frac{e_2}{2}, \frac{e_3}{2} \right\} \right\}$$

$$\text{Out[2544]} = \left\{ -f\left(\left\{\frac{e_1}{2}, \frac{e_6}{2}\right\}\right), f\left(\left\{\frac{e_2}{2}, \frac{e_3}{2}\right\}\right) \right\}$$

$$\text{Out[2545]} = \frac{e_5}{2}$$

In[2546]:= $\Lambda_7 \{-1, 1\}$ // StandardForm

$\Delta_{nm}[\Lambda_7] \{-1, 1\}$

Total@% /. f → fff

Out[2546]//StandardForm=

$$\left\{ \left\{ -\frac{\mathbf{i} e_1}{2}, -\frac{\mathbf{i} e_2}{2} \right\}, \left\{ \frac{\mathbf{i} e_3}{2}, \frac{\mathbf{i} e_6}{2} \right\} \right\}$$

$$\text{Out[2547]} = \left\{ -f\left(\left\{\frac{\mathbf{i} e_1}{2}, \frac{\mathbf{i} e_2}{2}\right\}\right), f\left(\left\{\frac{\mathbf{i} e_3}{2}, \frac{\mathbf{i} e_6}{2}\right\}\right) \right\}$$

$$\text{Out[2548]} = \frac{e_4}{2}$$

In[2549]:= $\Lambda_6 \Lambda_7 = (\Delta_{nm}[-\mathbf{i} \Lambda_6] + \Delta_{nm}[\Lambda_7]) \{-1, 1\}$

% /. f → fff

Total@%

$$\text{Out[2549]} = \left\{ -f\left(\left\{\frac{e_1}{2}, \frac{e_6}{2}\right\}\right) - f\left(\left\{\frac{\mathbf{i} e_1}{2}, \frac{\mathbf{i} e_2}{2}\right\}\right), f\left(\left\{\frac{e_2}{2}, \frac{e_3}{2}\right\}\right) + f\left(\left\{\frac{\mathbf{i} e_3}{2}, \frac{\mathbf{i} e_6}{2}\right\}\right) \right\}$$

$$\text{Out[2550]} = \left\{ \frac{e_4}{4} + \frac{e_5}{4}, \frac{e_4}{4} + \frac{e_5}{4} \right\}$$

$$\text{Out[2551]} = \frac{e_4}{2} + \frac{e_5}{2}$$

Use this change for 2014 u1 R→G fix:

In[2552]:= $\Lambda_6 \Lambda_7 = (**) - 2(*) 1 (**)$ ($\mathbf{i} \Delta_{nm}[\Lambda_6] + \Delta_{nm}[\Lambda_7]$) $\{-1, 1\}$

Expand@% /. f → fff

Total@%

$$\text{Out[2552]} = \left\{ 2 \left(f\left(\left\{\frac{\mathbf{i} e_1}{2}, \frac{\mathbf{i} e_2}{2}\right\}\right) + \mathbf{i} f\left(\left\{\frac{e_1}{2}, \frac{e_6}{2}\right\}\right) \right), -2 \left(\mathbf{i} f\left(\left\{\frac{e_2}{2}, \frac{e_3}{2}\right\}\right) + f\left(\left\{\frac{\mathbf{i} e_3}{2}, \frac{\mathbf{i} e_6}{2}\right\}\right) \right) \right\}$$

$$\text{Out[2553]} = \left\{ -\frac{e_4}{2} + \frac{\mathbf{i} e_5}{2}, -\frac{e_4}{2} + \frac{\mathbf{i} e_5}{2} \right\}$$

$$\text{Out[2554]} = -e_4 + \mathbf{i} e_5$$

In[2555]:= $(* \mathbf{u}1\mathbf{g} \text{ construction / deconstruction } *)$

In[2556]:= $\mathbf{i} \Lambda_4 \{1, -1\}$ // StandardForm

$\Delta_{nm}[\mathbf{i} \Lambda_4] \{1, -1\}$

Total@% /. f → fff

Out[2556]//StandardForm=

$$\left\{ \left\{ -\frac{e_2}{2}, -\frac{e_5}{2} \right\}, \left\{ \frac{e_4}{2}, \frac{e_6}{2} \right\} \right\}$$

$$\text{Out[2557]} = \left\{ f\left(\left\{-\frac{e_2}{2}, -\frac{e_5}{2}\right\}\right), -f\left(\left\{-\frac{e_4}{2}, -\frac{e_6}{2}\right\}\right) \right\}$$

$$\text{Out[2558]} = -\frac{e_3}{2}$$

In[2559]= $\Lambda_5 \{1, -1\}$ // StandardForm

$\Delta_{nm}[\Lambda_5] \{1, -1\}$
Total@% /. f → fff

Out[2559]//StandardForm=

$$\left\{ \left\{ -\frac{i e_5}{2}, -\frac{i e_6}{2} \right\}, \left\{ -\frac{i e_2}{2}, -\frac{i e_4}{2} \right\} \right\}$$

Out[2560]= $\left\{ f\left(\left\{-\frac{i e_5}{2}, -\frac{i e_6}{2}\right\}\right), -f\left(\left\{\frac{i e_2}{2}, \frac{i e_4}{2}\right\}\right) \right\}$

Out[2561]= 0

In[2562]= $\Lambda_4 \Lambda_5 = (\Delta_{nm}[i \Lambda_4] + \Delta_{nm}[\Lambda_5]) \{1, -1\}$

% /. f → fff
Total@%

Out[2562]= $\left\{ f\left(\left\{-\frac{e_2}{2}, -\frac{e_5}{2}\right\}\right) + f\left(\left\{-\frac{i e_5}{2}, -\frac{i e_6}{2}\right\}\right), -f\left(\left\{-\frac{e_4}{2}, -\frac{e_6}{2}\right\}\right) - f\left(\left\{\frac{i e_2}{2}, \frac{i e_4}{2}\right\}\right) \right\}$

Out[2563]= $\left\{ -\frac{e_1}{4} - \frac{e_3}{4}, \frac{e_1}{4} - \frac{e_3}{4} \right\}$

Out[2564]= $-\frac{e_3}{2}$

Use this change for 2014 u1 G→R fix:

In[2565]= $\Lambda_4 \Lambda_5 = (**) 2 (*) 4 (**)$ $(\Delta_{nm}[\Lambda_4] \{1, -1\} + i \Delta_{nm}[\Lambda_5])$

Expand[% /. f → fff]
Total@%

Out[2565]= $\left\{ 2\left(f\left(\left\{\frac{i e_2}{2}, \frac{i e_5}{2}\right\}\right) + i f\left(\left\{-\frac{i e_5}{2}, -\frac{i e_6}{2}\right\}\right)\right), 2\left(i f\left(\left\{\frac{i e_2}{2}, \frac{i e_4}{2}\right\}\right) - f\left(\left\{\frac{i e_4}{2}, \frac{i e_6}{2}\right\}\right)\right) \right\}$

Out[2566]= $\left\{ \frac{e_3}{2} - \frac{i e_1}{2}, \frac{e_3}{2} - \frac{i e_1}{2} \right\}$

Out[2567]= $e_3 - i e_1$

In[2568]= (* u1b construction / deconstruction *)

In[2569]= $i \Lambda_1 \{-1, 1\}$ // StandardForm

$\Delta_{nm}[i \Lambda_1] \{-1, 1\}$
Total@% /. f → fff

Out[2569]//StandardForm=

$$\left\{ \left\{ \frac{e_1}{2}, \frac{e_5}{2} \right\}, \left\{ \frac{e_3}{2}, \frac{e_4}{2} \right\} \right\}$$

Out[2570]= $\left\{ -f\left(\left\{-\frac{e_1}{2}, -\frac{e_5}{2}\right\}\right), f\left(\left\{\frac{e_3}{2}, \frac{e_4}{2}\right\}\right) \right\}$

Out[2571]= 0

In[2572]= $\Lambda_2 \{-1, 1\}$ // StandardForm

$\Delta_{nm}[\Lambda_2] \{-1, 1\}$
Total@% /. f → fff

Out[2572]//StandardForm=

$$\left\{ \left\{ \frac{i e_1}{2}, \frac{i e_4}{2} \right\}, \left\{ -\frac{i e_3}{2}, -\frac{i e_5}{2} \right\} \right\}$$

Out[2573]= $\left\{ -f\left(\left\{-\frac{i e_1}{2}, -\frac{i e_4}{2}\right\}\right), f\left(\left\{-\frac{i e_3}{2}, -\frac{i e_5}{2}\right\}\right) \right\}$

Out[2574]= $-\frac{e_2}{2}$

In[2575]= $\Lambda 1 \Lambda 2 = (\Lambda n m [\mathbf{i} \Lambda_1] + \Lambda n m [\Lambda_2]) \{-1, 1\}$

% /. f \rightarrow fff

Total@%

Out[2575]= $\left\{-f\left(\left\{-\frac{e_1}{2}, -\frac{e_5}{2}\right\}\right) - f\left(\left\{-\frac{i e_1}{2}, -\frac{i e_4}{2}\right\}\right), f\left(\left\{\frac{e_3}{2}, \frac{e_4}{2}\right\}\right) + f\left(\left\{-\frac{i e_3}{2}, -\frac{i e_5}{2}\right\}\right)\right\}$

Out[2576]= $\left\{-\frac{e_2}{4} - \frac{e_6}{4}, \frac{e_6}{4} - \frac{e_2}{4}\right\}$

Out[2577]= $-\frac{e_2}{2}$

In[2578]= (* u1 RGB *)

In[2579]= **u1r := u1[$\Lambda 6 \Lambda 7$];**

u1r

u1r /. f \rightarrow fff

Out[2580]= $\left\{\{s, S^*\}, \left\{2\left(f\left(\left\{\frac{i e_1}{2}, \frac{i e_2}{2}\right\}\right) + i f\left(\left\{\frac{i e_1}{2}, \frac{i e_6}{2}\right\}\right)\right) - 2\left(i f\left(\left\{\frac{i e_2}{2}, \frac{i e_3}{2}\right\}\right) + f\left(\left\{\frac{i e_3}{2}, \frac{i e_6}{2}\right\}\right)\right)\right\}, \{s, S\}\right\}$

Out[2581]= $\left\{\{s, S^*\}, \left\{2\left(-\frac{e_4}{4} + \frac{i e_5}{4}\right) - 2\left(\frac{e_4}{4} - \frac{i e_5}{4}\right)\right\}, \{s, S\}\right\}$

In[2582]= (* From 2014 paper *)

%[[2, 1]] $\circ v$

Out[2582]= $-e_4 + i e_5$

In[2583]= **u1g := u1[$\Lambda 4 \Lambda 5$];**

u1g

u1g /. f \rightarrow fff

Out[2584]= $\left\{\{s, S^*\}, \left\{2\left(i f\left(\left\{\frac{i e_2}{2}, \frac{i e_4}{2}\right\}\right) - f\left(\left\{\frac{i e_4}{2}, \frac{i e_6}{2}\right\}\right)\right) + 2\left(f\left(\left\{\frac{i e_2}{2}, \frac{i e_5}{2}\right\}\right) + i f\left(\left\{-\frac{i e_5}{2}, -\frac{i e_6}{2}\right\}\right)\right)\right\}, \{s, S\}\right\}$

Out[2585]= $\left\{\{s, S^*\}, \left\{4\left(\frac{e_3}{4} - \frac{i e_1}{4}\right)\right\}, \{s, S\}\right\}$

In[2586]= (* From 2014 paper *)

%[[2, 1]] $\circ v$

Out[2586]= 0

In[2587]= **u1b := u1[$\Lambda 1 \Lambda 2$];**

u1b

u1b /. f \rightarrow fff

Out[2588]= $\left\{\{s, S^*\}, \left\{-f\left(\left\{-\frac{e_1}{2}, -\frac{e_5}{2}\right\}\right) - f\left(\left\{-\frac{i e_1}{2}, -\frac{i e_4}{2}\right\}\right) + f\left(\left\{-\frac{i e_3}{2}, -\frac{i e_5}{2}\right\}\right) + f\left(\left\{\frac{e_3}{2}, \frac{e_4}{2}\right\}\right)\right\}, \{s, S\}\right\}$

Out[2589]= $\left\{\{s, S^*\}, \left\{-\frac{e_2}{2}\right\}, \{s, S\}\right\}$

In[2590]= (* From 2014 paper *)

%[[2, 1]] $\circ v$

Out[2590]= $-\frac{e_2}{4} + \frac{i e_6}{4}$

In[2591]= (* u2 RGB *)

In[2592]= **u2r := u2[Λ6Λ7];**

u2r

u2r /. f → fff

Out[2593]= $\left\{ \{s^*, S^*\}, \left\{ 2 \left(f \left(\left\{ \frac{i e_1}{2}, \frac{i e_2}{2} \right\} \right) + i f \left(\left\{ \frac{i e_1}{2}, \frac{i e_6}{2} \right\} \right) \right) - 2 \left(i f \left(\left\{ \frac{i e_2}{2}, \frac{i e_3}{2} \right\} \right) + f \left(\left\{ \frac{i e_3}{2}, \frac{i e_6}{2} \right\} \right) \right) \right\}, \{s^*, S\} \right\}$

Out[2594]= $\left\{ \{s^*, S^*\}, \left\{ 2 \left(-\frac{e_4}{4} + \frac{i e_5}{4} \right) - 2 \left(\frac{e_4}{4} - \frac{i e_5}{4} \right) \right\}, \{s^*, S\} \right\}$

In[2595]= **(* From 2014 paper *)**

%[[2, 1]] ◦ v

Out[2595]= $-e_4 + i e_5$

In[2596]= **u2g := u2[Λ4Λ5];**

u2g

u2g /. f → fff

Out[2597]= $\left\{ \{s^*, S^*\}, \left\{ 2 \left(i f \left(\left\{ \frac{i e_2}{2}, \frac{i e_4}{2} \right\} \right) - f \left(\left\{ \frac{i e_4}{2}, \frac{i e_6}{2} \right\} \right) \right) + 2 \left(f \left(\left\{ \frac{i e_2}{2}, \frac{i e_5}{2} \right\} \right) + i f \left(\left\{ -\frac{i e_5}{2}, -\frac{i e_6}{2} \right\} \right) \right) \right\}, \{s^*, S\} \right\}$

Out[2598]= $\left\{ \{s^*, S^*\}, \left\{ 4 \left(\frac{e_3}{4} - \frac{i e_1}{4} \right) \right\}, \{s^*, S\} \right\}$

In[2599]= **(* From 2014 paper *)**

%[[2, 1]] ◦ v

Out[2599]= 0

In[2600]= **u2b := u2[Λ1Λ2];**

u2b

u2b /. f → fff

Out[2601]= $\left\{ \{s^*, S^*\}, \left\{ -f \left(\left\{ -\frac{e_1}{2}, -\frac{e_5}{2} \right\} \right) - f \left(\left\{ -\frac{i e_1}{2}, -\frac{i e_4}{2} \right\} \right) + f \left(\left\{ -\frac{i e_3}{2}, -\frac{i e_5}{2} \right\} \right) + f \left(\left\{ \frac{e_3}{2}, \frac{e_4}{2} \right\} \right) \right\}, \{s^*, S\} \right\}$

Out[2602]= $\left\{ \{s^*, S^*\}, \left\{ -\frac{e_2}{2} \right\}, \{s^*, S\} \right\}$

In[2603]= **(* From 2014 paper *)**

%[[2, 1]] ◦ v

Out[2603]= $-\frac{e_2}{4} + \frac{i e_6}{4}$

In[2604]= **(* u3 RGB *)**

In[2605]= **u3r := u3[{ { f@{e1, e3, e4} + f@{e5}}, i {f@{e1, e3, e5} - f@{e4}} }];**

u3r

u3r /. f → fff

Out[2606]= $\{\{s, S\}, \{f(\{e_1, e_3, e_4\}) + i(f(\{e_1, e_3, e_5\}) - f(\{e_4\})) + f(\{e_5\})\}, \{s^*, S\}\}$

Out[2607]= $\{\{s, S\}, \{0\}, \{s^*, S\}\}$

In[2608]= **(* From 2014 paper *)**

%[[2, 1]] ◦ v

Out[2608]= 0

In[2609]= **{f@{e1, e3, e4},**

f@{e5},

i f@{e1, e3, e5},

-i f@{e4} } /. f → fff

Out[2609]= $\{-e_5, e_5, i e_4, -i e_4\}$

```

In[2610]= u3r := u3[ {{ f@{e1, e3, e4} + f@{e5} + i (f@{e1, e3, e5} - f@{e4})}}];
u3r
u3r /. f → fff

Out[2611]= {{s, S}, {f(e1, e3, e4) + i (f(e1, e3, e5) - f(e4)) + f(e5)}, {s*, S}}

Out[2612]= {{s, S}, {0}, {s*, S}}

In[2613]= (* From 2014 paper *)
%[[2, 1]] ◦ v

Out[2613]= 0

In[2614]= {f@{e1, e2, e6},
f@{e3},
f@{e1, e4, e5},
-i f@{e1}} /. f → fff

Out[2614]= {-e3, e3, -e3, -i e1}

In[2615]= u3g := u3[ {{ f@{e1, e2, e6} + f@{e3}}, {f@{e1, e4, e5} - i f@{e1}}];
u3g
u3g /. f → fff

Out[2616]= {{s, S}, {f(e1, e2, e6) + f(e1, e4, e5) - i f(e1) + f(e3)}, {s*, S}}

Out[2617]= {{s, S}, {-e3 - i e1}, {s*, S}}

In[2618]= (* From 2014 paper *)
%[[2, 1]] ◦ v

Out[2618]= -e3 - i e1

In[2619]= {-f@{e1, e2, e3},
f@{e6},
i f@{e1, e3, e6},
-i f@{e2}} /. f → fff

Out[2619]= {-e6, e6, i e2, -i e2}

In[2620]= u3b := u3[ {{ -f@{e1, e2, e3} + f@{e6}}, i {f@{e1, e3, e6} - f@{e2}}];
u3b
u3b /. f → fff

Out[2621]= {{s, S}, {-f(e1, e2, e3) + i (f(e1, e3, e6) - f(e2)) + f(e6)}, {s*, S}}

Out[2622]= {{s, S}, {0}, {s*, S}}

In[2623]= (* From 2014 paper *)
%[[2, 1]] ◦ v

Out[2623]= 0

In[2624]= (* d1 RGB *)

In[2625]= d123rgb@oct_ := (* Row@**) {sS[[2]], Total@oct, sS[[1]]};

In[2626]= d1r := d123rgb[ {{ f@{e1, e2, e6} - f@{e3}}, - {f@{e1, e4, e5} + i f@{e1}}];
d1r
d1r /. f → fff

Out[2627]= {{s, S*}, {f(e1, e2, e6) - f(e1, e4, e5) - i f(e1) - f(e3)}, {s*, S}}

Out[2628]= {{s, S*}, {-e3 - i e1}, {s*, S}}

```

In[2629]= **(* From 2014 paper *)**
 $\%[[2, 1]] \circ v$

Out[2629]= $-e_3 - i e_1$

In[2630]= **d1g := d123rgb[{{ f@{e1, e3, e4} + f@{e5}}, i{-f@{e1, e3, e5} + f@{e4}}}}];**
(* ~u3r *)
d1g
d1g /. f -> fff

Out[2631]= $\{\{s, S^*\}, \{f(\{e_1, e_3, e_4\}) + i(f(\{e_4\}) - f(\{e_1, e_3, e_5\})) + f(\{e_5\})\}, \{s^*, S\}\}$

Out[2632]= $\{\{s, S^*\}, \{0\}, \{s^*, S\}\}$

In[2633]= **(* From 2014 paper *)**
 $\%[[2, 1]] \circ v$

Out[2633]= 0

In[2634]= **d1b := d123rgb[{-f@{e1, e2, e5} + f@{e1, e4, e6}}, i{f@{e1, e5, e6} - f@{e1, e2, e4}}}}];**
d1b
d1b /. f -> fff

Out[2635]= $\{\{s, S^*\}, \{-f(\{e_1, e_2, e_5\}) - f(\{e_1, e_4, e_6\}) + i(f(\{e_1, e_5, e_6\}) - f(\{e_1, e_2, e_4\}))\}, \{s^*, S\}\}$

Out[2636]= $\{\{s, S^*\}, \{0\}, \{s^*, S\}\}$

In[2637]= **(* From 2014 paper *)**
 $\%[[2, 1]] \circ v$

Out[2637]= 0

In[2638]= **(* d2 RGB *)**

In[2639]= **d2r := d123rgb[{{ f@{e1, e2, e3} + f@{e6}}, i{f@{e1, e3, e6} + f@{e2}}}}];**
(* ~u3b *)
d2r
d2r /. f -> fff

Out[2640]= $\{\{s, S^*\}, \{f(\{e_1, e_2, e_3\}) + i(f(\{e_1, e_3, e_6\}) + f(\{e_2\})) + f(\{e_6\})\}, \{s^*, S\}\}$

Out[2641]= $\{\{s, S^*\}, \{2 e_6 + 2 i e_2\}, \{s^*, S\}\}$

In[2642]= **(* From 2014 paper *)**
 $\%[[2, 1]] \circ v$

Out[2642]= $2 e_6 + 2 i e_2$

In[2643]= **d2g := d123rgb[{-f@{e1, e2, e5} - f@{e1, e4, e6}}, -i{f@{e1, e5, e6} + f@{e1, e2, e4}}}}];**
(* ~d1b *)
d2g
d2g /. f -> fff

Out[2644]= $\{\{s, S^*\}, \{-f(\{e_1, e_2, e_5\}) + f(\{e_1, e_4, e_6\}) - i(f(\{e_1, e_2, e_4\}) + f(\{e_1, e_5, e_6\}))\}, \{s^*, S\}\}$

Out[2645]= $\{\{s, S^*\}, \{2 e_7 + 2 i\}, \{s^*, S\}\}$

In[2646]= **(* From 2014 paper *)**
 $\%[[2, 1]] \circ v$

Out[2646]= 0


```

In[2647]= d2b := d123rgb[{{ f@{e1, e3, e4} - f@{e5}}},          -i {f@{e1, e3, e5} + f@{e4}}}}];
          (* ~d1g *)
          d2b
          d2b /. f -> fff

Out[2648]= {{s, S*}, {f({e1, e3, e4}) - i (f({e1, e3, e5}) + f({e4})) - f({e5})}, {s*, S}}

Out[2649]= {{s, S*}, {-2 e5 - 2 i e4}, {s*, S}}

In[2650]= (* From 2014 paper *)
          %[[2, 1]] o v

Out[2650]= -2 e5 - 2 i e4

In[2651]= (* d3 RGB *)

In[2652]= d3r := d123rgb[{{ f@{e1, e2, e5} + f@{e1, e4, e6}}},          i {f@{e1, e5, e6} - f@{e1, e2, e4}}}}];
          (* ~d1b *)
          d3r
          d3r /. f -> fff

Out[2653]= {{s, S*}, {f({e1, e2, e5}) + f({e1, e4, e6}) + i (f({e1, e5, e6}) - f({e1, e2, e4}))}, {s*, S}}

Out[2654]= {{s, S*}, {0}, {s*, S}}

In[2655]= (* From 2014 paper *)
          %[[2, 1]] o v

Out[2655]= 0

In[2656]= d3g := d123rgb[{{ f@{e1, e2, e3} - f@{e6}}},          i {f@{e1, e3, e6} - f@{e2}}}}];
          (* ~u3b *)
          d3g
          d3g /. f -> fff

Out[2657]= {{s, S*}, {f({e1, e2, e3}) + i (f({e1, e3, e6}) - f({e2})) - f({e6})}, {s*, S}}

Out[2658]= {{s, S*}, {0}, {s*, S}}

In[2659]= (* From 2014 paper *)
          %[[2, 1]] o v

Out[2659]= 0

In[2660]= d3b := d123rgb[{{ f@{e1, e2, e6} + f@{e3}}},          - {f@{e1, e4, e5} - i f@{e1}}}}];
          (* ~d1r *)
          d3b
          d3b /. f -> fff

Out[2661]= {{s, S*}, {f({e1, e2, e6}) - f({e1, e4, e5}) + i f({e1}) + f({e3})}, {s*, S}}

Out[2662]= {{s, S*}, {e3 + i e1}, {s*, S}}

In[2663]= (* From 2014 paper *)
          %[[2, 1]] o v

Out[2663]= e3 + i e1

In[2664]= (* v *)

In[2665]= v1 := (* Row@**) {sS[[1]], Total@ {{1}},          i {f@{e1, e3} + f@{e2, e6} + f@{e4, e5}}}}, sS[[1]]};
          v1 /. f -> fff

Out[2666]= {{s*, S}, {1 + 3 i e7}, {s*, S}}

```

In[2667]= (* From 2014 paper *)

```
v1 := (* Row**) {sS[[1], {1 + i 0.}, sS[[1]]};
v1 /. f -> fff
%[[2, 1]] o v
```

Out[2668]= {{s*, S}, {1. + 0. i}, {s*, S}}

Out[2669]= 0.5 + (0. + 0.5 i) e7

In[2670]= (*

```
v1 *)
v2 := (* Row**) {sS[[2], Total@{{3}, -i {f@{e1, e3} + f@{e2, e6} + f@{e4, e5}}}, sS[[2]]};
v2 /. f -> fff
```

Out[2671]= {{s, S*}, {3 - 3 i e7}, {s, S*}}

In[2672]= (* From 2014 paper *)

```
(*
v1 *)
v2 := (* Row**) {sS[[2], Total@{{f@{e1, e3} + f@{e2, e6} + f@{e4, e5}}}, sS[[2]]};
v2 /. f -> fff
%[[2, 1]] o v
```

Out[2673]= {{s, S*}, {3 e7}, {s, S*}}

Out[2674]= $\frac{3 e7}{2} - \frac{3 i}{2}$

In[2675]= (*

```
d1b *)
v3 := (* Row**) {sS[[3], Total@{-f@{e1, e2, e5} + f@{e1, e4, e6}}, -i {f@{e1, e5, e6} + f@{e1, e2, e4}}}, sS[[4]]};
v3 /.
f ->
fff
```

Out[2676]= {{s*, S*}, {2 i}, {s, S}}

In[2677]= (* From 2014 paper *)

```
%[[2, 1]] o v
```

Out[2677]= -e7 + i

In[2678]= (* v *)

In[2679]= e123@oct_ := (* Row**) {sS[[2], Total@oct, sS[[3]]};

```
e1 := e123[ {{ f@{e1, e2, e6} - f@{e3}}, {f@{e1, e4, e5} + i f@{e1}} }];
(* ~d1r *)
e1 /. f -> fff
```

Out[2681]= {{s, S*}, {-3 e3 + i e1}, {s*, S*}}

In[2682]= (* From 2014 paper *)

```
e1 := e123[ {{ f@{e1, e2, e6} - f@{e3}}, {f@{e1, e4, e5} - i f@{e1}} }];
(* ~d1r *)
e1 /. f -> fff
%[[2, 1]] o v
```

Out[2683]= {{s, S*}, {-3 e3 - i e1}, {s*, S*}}

Out[2684]= -2 e3 - 2 i e1

```
In[2685]= e2 := e123[  {{-f@{e1, e2, e3} + f@{e6}},          - i {f@{e1, e3, e6} + f@{e2}}}}];
(* ~u3b *)
e2 /. f -> fff
```

```
Out[2686]= {{s, S*}, {-2 i e2}, {s*, S*}}
```

```
In[2687]= (* From 2014 paper *)
e2 := e123[  {{f@{e1, e2, e3} + f@{e6}},          i {f@{e1, e3, e6} + f@{e2}}}}];
(* ~u3b *)
e2 /. f -> fff
%[[2, 1]] o v
```

```
Out[2688]= {{s, S*}, {2 e6 + 2 i e2}, {s*, S*}}
```

```
Out[2689]= 2 e6 + 2 i e2
```

```
In[2690]= e3 := e123[  {{-f@{e1, e3, e4} + f@{e5}},          - i {f@{e1, e3, e5} + f@{e4}}}}];
(* ~u3r *)
e3 /. f -> fff
```

```
Out[2691]= {{s, S*}, {2 e5 - 2 i e4}, {s*, S*}}
```

```
In[2692]= (* From 2014 paper *)
e3 := e123[  {{-f@{e1, e3, e4} + f@{e5}},          i {f@{e1, e3, e5} + f@{e4}}}}];
(* ~u3r *)
e3 /. f -> fff
%[[2, 1]] o v
```

```
Out[2693]= {{s, S*}, {2 e5 + 2 i e4}, {s*, S*}}
```

```
Out[2694]= 2 e5 + 2 i e4
```

Define SU(3) Q to match the action in eqs. 19, 22 and 23

While I've tried many variations, this one is simply based on the position in the list of sS conjugate combinations (given the leading sS pair in the defined particle).

```
In[2695]= su3Q@n_ := Module[{in = n[[2]],
  pos = position[sS, n[[1]]]},
  (**)Simplify@(**)If[pos == {}, in, {(**)i, (**)  $\frac{pos}{3}$ , n[[1, 1]]} (*) / .
  {s ->  $\frac{1}{2}$  (f@{in} +  $\frac{i}{2}$  (-f@{e7, in} + f@{e1, e3, in} + f@{e2, e6, in} + f@{e4, e5, in})),
  s ->  $\frac{1}{2}$  (f@{in} +  $\frac{i}{2}$  (-f@{e7, in}, True) +
  f@{e1, e3, in}, True) + f@{e2, e6, in}, True) + f@{e4, e5, in}, True)}}];
```

This works for the example given in u1r,

but I can't figure out how to make it work across all flavors and generations given their particle definitions above. My confusion seems to be around the proper use of the sS components.

Do an example, showing the decomposition of sS and show su3Q taking u1r as input (with and without evaluating f):

```
In[2696]= doConjugate = True;
```

$$\text{In[2697]:= } \mathbf{sS} /. \left\{ \mathbf{s} \rightarrow \frac{1}{2} \left(\mathbf{f}@\{\mathbf{in}\} + \frac{i}{2} \left(-\mathbf{f}@\{e_7, \mathbf{in}\} + \mathbf{f}@\{e_1, e_3, \mathbf{in}\} + \mathbf{f}@\{e_2, e_6, \mathbf{in}\} + \mathbf{f}@\{e_4, e_5, \mathbf{in}\} \right) \right), \right. \\ \left. \mathbf{s} \rightarrow \frac{1}{2} \left(\mathbf{f}@\{\mathbf{in}\} + \frac{i}{2} \left(-\mathbf{f}[\{e_7, \mathbf{in}\}, \text{True}] + \mathbf{f}[\{e_1, e_3, \mathbf{in}\}, \text{True}] + \right. \right. \right. \\ \left. \left. \left. \mathbf{f}[\{e_2, e_6, \mathbf{in}\}, \text{True}] + \mathbf{f}[\{e_4, e_5, \mathbf{in}\}, \text{True}] \right) \right) \right\} /. \{\mathbf{in} \rightarrow \mathbf{1}, \mathbf{f} \rightarrow \mathbf{fff}\}$$

$$\text{Out[2697]= } \begin{pmatrix} \frac{1}{2}(1 - i e_7) & \frac{1}{2}(i e_7 + 1) \\ \frac{1}{2}(i e_7 + 1) & \frac{1}{2}(1 - i e_7) \\ \frac{1}{2}(1 - i e_7) & \frac{1}{2}(1 - i e_7) \\ \frac{1}{2}(i e_7 + 1) & \frac{1}{2}(i e_7 + 1) \end{pmatrix}$$

In[2698]:= **Row@u1r**

$$\text{Out[2698]= } \{s, S^s\} \left\{ 2 \left(f \left(\left\{ \frac{i e_1}{2}, \frac{i e_2}{2} \right\} \right) + i f \left(\left\{ \frac{i e_1}{2}, \frac{i e_6}{2} \right\} \right) \right) - 2 \left(i f \left(\left\{ \frac{i e_2}{2}, \frac{i e_3}{2} \right\} \right) + f \left(\left\{ \frac{i e_3}{2}, \frac{i e_6}{2} \right\} \right) \right) \right\} \{s, S\}$$

In[2699]:= **u1r[[2, 1]] /. f -> fff**

$$\text{Out[2699]= } 2 \left(-\frac{e_4}{4} + \frac{i e_5}{4} \right) - 2 \left(\frac{e_4}{4} - \frac{i e_5}{4} \right)$$

In[2700]:= **Expand@** $\frac{\#}{3}$ **{f@{sS[[#, 1]]}, f@{sS[[#, 2]]}}** /.

$$\left\{ \mathbf{s} \rightarrow \frac{1}{2} \left(\mathbf{f}@\{\mathbf{in}\} + \frac{i}{2} \left(-\mathbf{f}@\{e_7, \mathbf{in}\} + \mathbf{f}@\{e_1, e_3, \mathbf{in}\} + \mathbf{f}@\{e_2, e_6, \mathbf{in}\} + \mathbf{f}@\{e_4, e_5, \mathbf{in}\} \right) \right), \right. \\ \left. \mathbf{s} \rightarrow \frac{1}{2} \left(\mathbf{f}@\{\mathbf{in}\} + \frac{i}{2} \left(-\mathbf{f}[\{e_7, \mathbf{in}\}, \text{True}] + \mathbf{f}[\{e_1, e_3, \mathbf{in}\}, \text{True}] + \right. \right. \right. \\ \left. \left. \left. \mathbf{f}[\{e_2, e_6, \mathbf{in}\}, \text{True}] + \mathbf{f}[\{e_4, e_5, \mathbf{in}\}, \text{True}] \right) \right) \right\} /. \\ \{\mathbf{in} \rightarrow \mathbf{u1r}[[2, 1]] /. \mathbf{f} \rightarrow \mathbf{fff}\} \& /@ \text{Range@3}$$

% /. f -> fff

Expand@Total[Total/@%]

$$\text{Out[2700]= } \begin{pmatrix} \frac{1}{3} f \left(\left\{ \frac{1}{2} \left(f \left(\left\{ 2 \left(\frac{i e_5}{4} - \frac{e_4}{4} \right) - 2 \left(\frac{e_4}{4} - \frac{i e_5}{4} \right) \right\} \right) + \frac{1}{2} i \left(-f \left(\left\{ e_7, 2 \left(\frac{i e_5}{4} - \frac{e_4}{4} \right) - 2 \left(\frac{e_4}{4} - \frac{i e_5}{4} \right) \right\} \right) + f \left(\left\{ e_1, e_3, 2 \left(\frac{i e_5}{4} - \frac{e_4}{4} \right) - 2 \left(\frac{e_4}{4} - \frac{i e_5}{4} \right) \right\} \right) + f \left(\left\{ e_2, e_6, 2 \left(\frac{i e_5}{4} - \frac{e_4}{4} \right) - 2 \left(\frac{e_4}{4} - \frac{i e_5}{4} \right) \right\} \right) \right) \right. \\ \left. \frac{2}{3} f \left(\left\{ \frac{1}{2} \left(f \left(\left\{ 2 \left(\frac{i e_5}{4} - \frac{e_4}{4} \right) - 2 \left(\frac{e_4}{4} - \frac{i e_5}{4} \right) \right\} \right) + \frac{1}{2} i \left(-f \left(\left\{ e_7, 2 \left(\frac{i e_5}{4} - \frac{e_4}{4} \right) - 2 \left(\frac{e_4}{4} - \frac{i e_5}{4} \right) \right\} \right) + f \left(\left\{ e_1, e_3, 2 \left(\frac{i e_5}{4} - \frac{e_4}{4} \right) - 2 \left(\frac{e_4}{4} - \frac{i e_5}{4} \right) \right\} \right) + f \left(\left\{ e_2, e_6, 2 \left(\frac{i e_5}{4} - \frac{e_4}{4} \right) - 2 \left(\frac{e_4}{4} - \frac{i e_5}{4} \right) \right\} \right) \right) \right. \right. \\ \left. \left. f \left(\left\{ \frac{1}{2} \left(f \left(\left\{ 2 \left(\frac{i e_5}{4} - \frac{e_4}{4} \right) - 2 \left(\frac{e_4}{4} - \frac{i e_5}{4} \right) \right\} \right) + \frac{1}{2} i \left(-f \left(\left\{ e_7, 2 \left(\frac{i e_5}{4} - \frac{e_4}{4} \right) - 2 \left(\frac{e_4}{4} - \frac{i e_5}{4} \right) \right\} \right) + f \left(\left\{ e_1, e_3, 2 \left(\frac{i e_5}{4} - \frac{e_4}{4} \right) - 2 \left(\frac{e_4}{4} - \frac{i e_5}{4} \right) \right\} \right) + f \left(\left\{ e_2, e_6, 2 \left(\frac{i e_5}{4} - \frac{e_4}{4} \right) - 2 \left(\frac{e_4}{4} - \frac{i e_5}{4} \right) \right\} \right) \right) \right) \right. \right. \end{pmatrix}$$

$$\text{Out[2701]= } \begin{pmatrix} \frac{1}{3}(-e_4 - i e_5) & 0 \\ \frac{2}{3}(i e_5 - e_4) & 0 \\ -e_4 - i e_5 & 0 \end{pmatrix}$$

$$\text{Out[2702]= } -2 e_4 - \frac{2 i e_5}{3}$$

In[2703]:= **doConjugate = False;**
su3Q@u1r[[2, 1]]

$$\text{Out[2704]= } -2 \left(i f \left(\left\{ \frac{i e_2}{2}, \frac{i e_3}{2} \right\} \right) + f \left(\left\{ \frac{i e_3}{2}, \frac{i e_6}{2} \right\} \right) \right)$$

In[2705]:= **doConjugate = True;**
su3Q@u1r[[2, 1]]

$$\text{Out[2706]= } -2 \left(i f \left(\left\{ \frac{i e_2}{2}, \frac{i e_3}{2} \right\} \right) + f \left(\left\{ \frac{i e_3}{2}, \frac{i e_6}{2} \right\} \right) \right)$$

From the 2014 particle model in [2] show commutation action for U1R→U1G with octonion multiplication

Cohl doesn't explicitly reduce the octonion elements via octonionic multiplication here, but does reduce them via common identities. Here we process the manual reduction of the color changing action of $\Lambda_1[\nu] \odot \mathbf{u1r} = \nu \circ \mathbf{u1g}$

Now test the reverse action ($\Lambda 1[\nu] \odot u1g[[2, 1]]$, for which we need to use ν (not $-\nu^*$ as above) which would drive this result to zero):

```
In[2722]:= (* u1 G→R *)
(Λ[1, ν] /. f → fff) ⊙ (u1g[[2, 1]] /. f → fff)
u1r[[2, 1]] /. f → fff
Expand[% == %%]
```

```
Out[2722]:= -e4 + i e5
```

```
Out[2723]:= 2(-e4/4 + i e5/4) - 2(e4/4 - i e5/4)
```

```
Out[2724]:= True
```

Moving on, we now build the machinery to perform these commutations across the SM particles:

```
In[2725]:= (* Function to output the action with input values of Λnν and the bioctonion particle *)
doAction[Λg_, particle_] := Expand[(Λg /. f → fff) ⊙ (particle /. f → fff)];
```

```
In[2726]:= (* u1 R→G *)
```

```
In[2727]:= doAction[Λ[1, -ν*], u1r[[2, 1]]]
u1g[[2, 1]] /. f → fff
Expand[% == %%]
```

```
Out[2727]:= e3 - i e1
```

```
Out[2728]:= 4(e3/4 - i e1/4)
```

```
Out[2729]:= True
```

```
In[2730]:= (* u1 G→R *)
```

```
In[2731]:= doAction[Λ1[ν], u1g[[2, 1]]]
u1r[[2, 1]] /. f → fff
Expand[% == %%]
```

```
Out[2731]:= -e4 + i e5
```

```
Out[2732]:= 2(-e4/4 + i e5/4) - 2(e4/4 - i e5/4)
```

```
Out[2733]:= True
```

Function to output the action across SM particles with a string input identifying Λ_n .

I need more information on the flavor and color transformations involved in Λ_n , but this helps deduce that by trying various combinations.

```
In[2734]:= (* Define the list of particle flavors *)
flavorList = Flatten@pTypeList[; ; 2]
```

```
Out[2734]:= {e, ν, d, u}
```

```
In[2735]= doSMaction@in_ := Column[Table[Column[Table[Row[Table[
  inExp = ToExpression@in;
  prt = ToExpression[flavorList[[flv]] <> ToString@gen <> If[flv > 2, rgb[[col]], " "]];
  sprt = ToString[
    Subsuperscript[flavorList[[flv]], gen, If[flv > 2, rgb[[col]], " "], TraditionalForm];
  ssprt = Style[sprt, Bold, Red];
  (* skip color iterations on leptons *)
  If[flv < 3 && col > 1, Nothing,
    Column[{{ssprt, Row@{sprt <> "= ", Row@{Expand@prt[[2, 1]] /. f -> fff}, " "}},
      Row@{in <> "O" <> sprt <> "= ",
        doAction[inExp, prt[[2, 1]] /. f -> fff}, " "}}, Center]],
    {col, 3}]],
  {flv, 4}], Center],
  {gen, 3}], Center];
```

```
In[2736]= (* u1 R→G *)
doSMaction@"Λ[1,-ν*]"
```

$$\begin{aligned} & \mathbf{e}_1 \\ e_1 &= -3e_3 - ie_1 \\ \Lambda[1,-\nu^*] \odot e_1 &= -2e_4 - 2ie_5 \end{aligned}$$

$$\begin{aligned} & \mathbf{v}_1 \\ v_1 &= 1. + 0. i \\ \Lambda[1,-\nu^*] \odot v_1 &= 0 \end{aligned}$$

$$\begin{aligned} & \mathbf{d}_1^r & & \mathbf{d}_1^g & & \mathbf{d}_1^b \\ d_1^r &= -e_3 - ie_1 & & d_1^g &= 0 & & d_1^b &= 0 \\ \Lambda[1,-\nu^*] \odot d_1^r &= -e_4 - ie_5 & & \Lambda[1,-\nu^*] \odot d_1^g &= 0 & & \Lambda[1,-\nu^*] \odot d_1^b &= 0 \end{aligned}$$

$$\begin{aligned} & \mathbf{u}_1^r & & \mathbf{u}_1^g & & \mathbf{u}_1^b \\ u_1^r &= -e_4 + ie_5 & & u_1^g &= e_3 - ie_1 & & u_1^b &= -\frac{e_2}{2} \\ \Lambda[1,-\nu^*] \odot u_1^r &= e_3 - ie_1 & & \Lambda[1,-\nu^*] \odot u_1^g &= 0 & & \Lambda[1,-\nu^*] \odot u_1^b &= \frac{e_7}{4} \end{aligned}$$

$$\begin{aligned} & \mathbf{e}_2 \\ e_2 &= 2e_6 + 2ie_2 \\ \Lambda[1,-\nu^*] \odot e_2 &= 0 \end{aligned}$$

$$\begin{aligned} & \mathbf{v}_2 \\ v_2 &= 3e_7 \\ \Lambda[1,-\nu^*] \odot v_2 &= \frac{3e_2}{2} - \frac{3ie_6}{2} \end{aligned}$$

$$\begin{aligned} & \mathbf{d}_2^r & & \mathbf{d}_2^g & & \mathbf{d}_2^b \\ d_2^r &= 2e_6 + 2ie_2 & & d_2^g &= 2e_7 + 2i & & d_2^b &= -2e_5 - 2ie_4 \\ \Lambda[1,-\nu^*] \odot d_2^r &= 0 & & \Lambda[1,-\nu^*] \odot d_2^g &= e_2 - ie_6 & & \Lambda[1,-\nu^*] \odot d_2^b &= 2e_1 + 2ie_3 \end{aligned}$$

$$\begin{aligned} & \mathbf{u}_2^r & & \mathbf{u}_2^g & & \mathbf{u}_2^b \\ u_2^r &= -e_4 + ie_5 & & u_2^g &= e_3 - ie_1 & & u_2^b &= -\frac{e_2}{2} \\ \Lambda[1,-\nu^*] \odot u_2^r &= e_3 - ie_1 & & \Lambda[1,-\nu^*] \odot u_2^g &= 0 & & \Lambda[1,-\nu^*] \odot u_2^b &= \frac{e_7}{4} \end{aligned}$$

$$\begin{aligned} & \mathbf{e}_3 \\ e_3 &= 2e_5 + 2ie_4 \\ \Lambda[1,-\nu^*] \odot e_3 &= -2e_1 - 2ie_3 \end{aligned}$$

$$\begin{aligned} & \mathbf{v}_3 \\ v_3 &= 2i \\ \Lambda[1,-\nu^*] \odot v_3 &= 0 \end{aligned}$$

$$\begin{aligned} & \mathbf{d}_3^r & & \mathbf{d}_3^g & & \mathbf{d}_3^b \\ d_3^r &= 0 & & d_3^g &= 0 & & d_3^b &= e_3 + ie_1 \\ \Lambda[1,-\nu^*] \odot d_3^r &= 0 & & \Lambda[1,-\nu^*] \odot d_3^g &= 0 & & \Lambda[1,-\nu^*] \odot d_3^b &= e_4 + ie_5 \end{aligned}$$

$$\begin{aligned} & \mathbf{u}_3^r & & \mathbf{u}_3^g & & \mathbf{u}_3^b \\ u_3^r &= 0 & & u_3^g &= -e_3 - ie_1 & & u_3^b &= 0 \\ \Lambda[1,-\nu^*] \odot u_3^r &= 0 & & \Lambda[1,-\nu^*] \odot u_3^g &= -e_4 - ie_5 & & \Lambda[1,-\nu^*] \odot u_3^b &= 0 \end{aligned}$$

Out[2736]=

Visualize the entire family of flavors across 3 generations (2014 version)

These show the unevaluated form as well as the octonion reduced form, with the unevaluated (string form) showing the appending of "o v" along with the evaluation of that:

```
In[2737]:= doSM14 := Column[Table[Column[Table[Row[Table[
  prt = Expand[ToExpression[flavorList[[flv]] <> ToString@gen <> If[flv > 2, rgb[[col]], " "]];
  sprt = ToString[
    Subsuperscript[flavorList[[flv]], gen, If[flv > 2, rgb[[col]], " "], TraditionalForm];
  ssprt = Style[sprt, Bold, Red];
  sprt1 = ToString[Expand[prt[[2, 1]], TraditionalForm] <> " ";
  sprt2 = ToString[Expand[prt[[2, 1]] /. f -> fff], TraditionalForm];
  sprt3 = sprt <> "o v=" <> ToString[Expand[(prt[[2, 1]] /. f -> fff) o v], TraditionalForm];
  (* skip color iterations on leptons *)
  If[flv < 3 && col > 1, Nothing,
    Column[{ssprt, sprt1, sprt2, sprt3}, Center]],
  {col, 3}]],
  {flv, 4}], Center],
  {gen, 3}], Center];
```

In[2738]:= doSM14

$$\begin{array}{c}
 \begin{array}{c}
 \mathbf{e}_1 \\
 f(e_1, e_2, e_6) + f(e_1, e_4, e_5) - i f(e_1) - f(e_3) \\
 -3e_3 - i e_1 \\
 e_1 \circ v = -2e_3 - 2i e_1 \\
 \mathbf{v}_1 \\
 1 + 0i \\
 \mathbf{v}_1 \circ v = 0.5 + (0 + 0.5i) e_7
 \end{array} \\
 \begin{array}{ccc}
 \mathbf{d}_f & \mathbf{d}_f & \mathbf{d}_f \\
 f(e_1, e_2, e_6) - f(e_1, e_4, e_5) - i f(e_1) - f(e_3) & f(e_1, e_3, e_4) - i f(e_1, e_3, e_5) + i f(e_4) + f(e_5) & -i f(e_1, e_2, e_4) - f(e_1, e_2, e_5) - f(e_1, e_4, e_6) + i f(e_1, e_5, e_6) \\
 -e_3 - i e_1 & \mathbf{d}_f \circ v = 0 & \mathbf{d}_f \circ v = 0
 \end{array} \\
 \begin{array}{ccc}
 \mathbf{u}_f & \mathbf{u}_f & \mathbf{u}_f \\
 2f\left(\frac{i e_4 + i e_5}{2}\right) + 2i f\left(\frac{i e_4 + i e_5}{2}\right) - 2f\left(\frac{i e_3 + i e_6}{2}\right) & 2i f\left(\frac{i e_4 + i e_5}{2}\right) + 2f\left(\frac{i e_3 + i e_6}{2}\right) - 2f\left(\frac{i e_4 + i e_5}{2}\right) + 2i f\left(\frac{-i e_3 - i e_6}{2}\right) & -f\left(\frac{-e_4 - i e_5}{2}\right) - f\left(\frac{-i e_4 - i e_5}{2}\right) + f\left(\frac{-i e_3 - i e_6}{2}\right) + f\left(\frac{e_3 + e_6}{2}\right) \\
 -e_4 + i e_5 & e_3 - i e_1 & -\frac{e_2}{2} \\
 \mathbf{u}_f \circ v = -e_4 + i e_5 & \mathbf{u}_f \circ v = 0 & \mathbf{u}_f \circ v = -\frac{e_2}{4} + \frac{i e_6}{4}
 \end{array} \\
 \begin{array}{c}
 \mathbf{e}_2 \\
 f(e_1, e_2, e_3) + i f(e_1, e_3, e_6) + i f(e_2) + f(e_6) \\
 2e_6 + 2i e_2 \\
 e_2 \circ v = 2e_6 + 2i e_2 \\
 \mathbf{v}_2 \\
 3e_7 \\
 \mathbf{v}_2 \circ v = \frac{3e_7}{2} - \frac{3i}{2}
 \end{array} \\
 \begin{array}{ccc}
 \mathbf{d}_g & \mathbf{d}_g & \mathbf{d}_g \\
 f(e_1, e_2, e_3) + i f(e_1, e_3, e_6) + i f(e_2) + f(e_6) & -i f(e_1, e_2, e_4) - f(e_1, e_2, e_5) + f(e_1, e_4, e_6) - i f(e_1, e_5, e_6) & f(e_1, e_3, e_4) - i f(e_1, e_3, e_5) - i f(e_4) - f(e_5) \\
 2e_6 + 2i e_2 & \mathbf{d}_g \circ v = 0 & \mathbf{d}_g \circ v = -2e_5 - 2i e_4
 \end{array} \\
 \begin{array}{ccc}
 \mathbf{u}_g & \mathbf{u}_g & \mathbf{u}_g \\
 2f\left(\frac{i e_4 + i e_5}{2}\right) + 2i f\left(\frac{i e_4 + i e_5}{2}\right) - 2f\left(\frac{i e_3 + i e_6}{2}\right) & 2i f\left(\frac{i e_4 + i e_5}{2}\right) + 2f\left(\frac{i e_3 + i e_6}{2}\right) - 2f\left(\frac{i e_4 + i e_5}{2}\right) + 2i f\left(\frac{-i e_3 - i e_6}{2}\right) & -f\left(\frac{-e_4 - i e_5}{2}\right) - f\left(\frac{-i e_4 - i e_5}{2}\right) + f\left(\frac{-i e_3 - i e_6}{2}\right) + f\left(\frac{e_3 + e_6}{2}\right) \\
 -e_4 + i e_5 & e_3 - i e_1 & -\frac{e_2}{2} \\
 \mathbf{u}_g \circ v = -e_4 + i e_5 & \mathbf{u}_g \circ v = 0 & \mathbf{u}_g \circ v = -\frac{e_2}{4} + \frac{i e_6}{4}
 \end{array} \\
 \begin{array}{c}
 \mathbf{e}_3 \\
 -f(e_1, e_3, e_4) + i f(e_1, e_3, e_5) + i f(e_4) + f(e_5) \\
 2e_5 + 2i e_4 \\
 e_3 \circ v = 2e_5 + 2i e_4 \\
 \mathbf{v}_3 \\
 2i \\
 \mathbf{v}_3 \circ v = -e_7 + i
 \end{array} \\
 \begin{array}{ccc}
 \mathbf{d}_h & \mathbf{d}_h & \mathbf{d}_h \\
 -i f(e_1, e_2, e_4) + f(e_1, e_2, e_5) + f(e_1, e_4, e_6) + i f(e_1, e_5, e_6) & f(e_1, e_2, e_3) + i f(e_1, e_3, e_6) - i f(e_2) - f(e_6) & f(e_1, e_2, e_6) - f(e_1, e_4, e_5) + i f(e_1) + f(e_3) \\
 \mathbf{d}_h \circ v = 0 & \mathbf{d}_h \circ v = 0 & \mathbf{d}_h \circ v = e_3 + i e_1
 \end{array} \\
 \begin{array}{ccc}
 \mathbf{u}_h & \mathbf{u}_h & \mathbf{u}_h \\
 f(e_1, e_3, e_4) + i f(e_1, e_3, e_5) - i f(e_4) + f(e_5) & f(e_1, e_2, e_6) + f(e_1, e_4, e_5) - i f(e_1) + f(e_3) & -f(e_1, e_2, e_3) + i f(e_1, e_3, e_6) - i f(e_2) + f(e_6) \\
 \mathbf{u}_h \circ v = 0 & \mathbf{u}_h \circ v = -e_3 - i e_1 & \mathbf{u}_h \circ v = 0
 \end{array}
 \end{array}$$

Visualize the entire family of flavors across 3 generations (2019 version)

These show the reduced octonion structure with pre/post sS, the Q based on the sS used (not sure if this is what is intended), as well as the the unevaluated (string form) showing what I assume is the commutation structure of eqs. 19, 22 and 23 action to derive proper charge Q (which doesn't seem to work at all). I need clarification on how this is supposed to operate (maybe more examples):

```
In[2739]:= doSM := Column[Table[Column[Table[Row[Table[
  prt = Expand[
    ToExpression[flavorList[[flv]] <> ToString@gen <> If[flv > 2, rgb[[col], " "]] /. f -> fff];
  sprt = ToString[Subsuperscript[flavorList[[flv]], gen, If[flv > 2, rgb[[col], " "]],
    TraditionalForm];
  ssprt = Style[sprt, Bold, Red];
  sQ = su3Q@prt;
  sQ1 = "Q=" <> ToString[sQ[[2]], TraditionalForm];
  sQ2 = ToString[Row@sQ, TraditionalForm] <> "⊙" <> ToString[prt[[1, 2]], TraditionalForm] <>
    sprt <> ToString[prt[[1, 1]], TraditionalForm] <> "=";
  sQ3 = "i (" <> ToString[sQ[[2]], TraditionalForm] <> ToString[Row@prt[[1], TraditionalForm] <>
    "⊙" <> ToString[Row@prt[[1]], TraditionalForm] <> sprt <>
    ToString[Row@prt[[3]], TraditionalForm] <> ")=";
  sQ4 = ToString[Row@sQ[[ ; 2]], TraditionalForm] <> sprt;
  (* skip color iterations on leptons *)
  If[flv < 3 && col > 1, Nothing,
    Column[{ssprt, Row@prt, sQ1, Row@{sQ2, sQ3, sQ4, " "}}, Center]],
    {col, 3}]],
  {flv, 4}], Center],
  {gen, 3}], Center];
```

In[2740]= doSM

$$\begin{aligned} & \mathbf{e}_1 \\ & \{s, S^*\} \{-3 e_3 - i e_1\} \{s^*, S^*\} \\ & Q = \frac{2}{3} \end{aligned}$$

$$i \frac{2}{3} s \odot S^* e_1 s = i \left(\frac{2}{3} s S^* \odot s S^* e_1 s^* S^* \right) = i \frac{2}{3} e_1$$

$$\begin{aligned} & \mathbf{v}_1 \\ & \{s^*, S\} \{1. + 0. i\} \{s^*, S\} \\ & Q = \frac{1}{3} \end{aligned}$$

$$i \frac{1}{3} s^* \odot S v_1 s^* = i \left(\frac{1}{3} s^* S \odot s^* S v_1 s^* S \right) = i \frac{1}{3} v_1$$

$$\begin{aligned} & \mathbf{d}_1^f \\ & \{s, S^*\} \{-e_3 - i e_1\} \{s^*, S\} \\ & Q = \frac{2}{3} \end{aligned}$$

$$i \frac{2}{3} s \odot S^* d_1^f s = i \left(\frac{2}{3} s S^* \odot s S^* d_1^f s^* S \right) = i \frac{2}{3} d_1^f$$

$$\begin{aligned} & \mathbf{d}_1^g \\ & \{s, S^*\} \{0\} \{s^*, S\} \\ & Q = \frac{2}{3} \end{aligned}$$

$$i \frac{2}{3} s \odot S^* d_1^g s = i \left(\frac{2}{3} s S^* \odot s S^* d_1^g s^* S \right) = i \frac{2}{3} d_1^g$$

$$\begin{aligned} & \mathbf{d}_1^h \\ & \{s, S^*\} \{0\} \{s^*, S\} \\ & Q = \frac{2}{3} \end{aligned}$$

$$i \frac{2}{3} s \odot S^* d_1^h s = i \left(\frac{2}{3} s S^* \odot s S^* d_1^h s^* S \right) = i \frac{2}{3} d_1^h$$

$$\begin{aligned} & \mathbf{u}_1^f \\ & \{s, S^*\} \{-e_4 + i e_5\} \{s, S\} \\ & Q = \frac{2}{3} \end{aligned}$$

$$i \frac{2}{3} s \odot S^* u_1^f s = i \left(\frac{2}{3} s S^* \odot s S^* u_1^f s S \right) = i \frac{2}{3} u_1^f$$

$$\begin{aligned} & \mathbf{u}_1^g \\ & \{s, S^*\} \{e_3 - i e_1\} \{s, S\} \\ & Q = \frac{2}{3} \end{aligned}$$

$$i \frac{2}{3} s \odot S^* u_1^g s = i \left(\frac{2}{3} s S^* \odot s S^* u_1^g s S \right) = i \frac{2}{3} u_1^g$$

$$\begin{aligned} & \mathbf{u}_1^h \\ & \{s, S^*\} \left\{ -\frac{e_2}{2} \right\} \{s, S\} \\ & Q = \frac{2}{3} \end{aligned}$$

$$i \frac{2}{3} s \odot S^* u_1^h s = i \left(\frac{2}{3} s S^* \odot s S^* u_1^h s S \right) = i \frac{2}{3} u_1^h$$

$$\begin{aligned} & \mathbf{e}_2 \\ & \{s, S^*\} \{2 e_6 + 2 i e_2\} \{s^*, S^*\} \\ & Q = \frac{2}{3} \end{aligned}$$

$$i \frac{2}{3} s \odot S^* e_2 s = i \left(\frac{2}{3} s S^* \odot s S^* e_2 s^* S^* \right) = i \frac{2}{3} e_2$$

$$\begin{aligned} & \mathbf{v}_2 \\ & \{s, S^*\} \{3 e_7\} \{s, S^*\} \\ & Q = \frac{2}{3} \end{aligned}$$

$$i \frac{2}{3} s \odot S^* v_2 s = i \left(\frac{2}{3} s S^* \odot s S^* v_2 s S^* \right) = i \frac{2}{3} v_2$$

$$\begin{aligned} & \mathbf{d}_2^f \\ & \{s, S^*\} \{2 e_6 + 2 i e_2\} \{s^*, S\} \\ & Q = \frac{2}{3} \end{aligned}$$

$$i \frac{2}{3} s \odot S^* d_2^f s = i \left(\frac{2}{3} s S^* \odot s S^* d_2^f s^* S \right) = i \frac{2}{3} d_2^f$$

$$\begin{aligned} & \mathbf{d}_2^g \\ & \{s, S^*\} \{2 e_7 + 2 i\} \{s^*, S\} \\ & Q = \frac{2}{3} \end{aligned}$$

$$i \frac{2}{3} s \odot S^* d_2^g s = i \left(\frac{2}{3} s S^* \odot s S^* d_2^g s^* S \right) = i \frac{2}{3} d_2^g$$

$$\begin{aligned} & \mathbf{d}_2^h \\ & \{s, S^*\} \{-2 e_5 - 2 i e_4\} \{s^*, S\} \\ & Q = \frac{2}{3} \end{aligned}$$

$$i \frac{2}{3} s \odot S^* d_2^h s = i \left(\frac{2}{3} s S^* \odot s S^* d_2^h s^* S \right) = i \frac{2}{3} d_2^h$$

$$\begin{aligned} & \mathbf{u}_2^f \\ & \{s^*, S^*\} \{-e_4 + i e_5\} \{s^*, S\} \\ & Q = 1 \end{aligned}$$

$$i 1 s^* \odot S^* u_2^f s^* = i \left(1 s^* S^* \odot s^* S^* u_2^f s^* S \right) = i 1 u_2^f$$

$$\begin{aligned} & \mathbf{u}_2^g \\ & \{s^*, S^*\} \{e_3 - i e_1\} \{s^*, S\} \\ & Q = 1 \end{aligned}$$

$$i 1 s^* \odot S^* u_2^g s^* = i \left(1 s^* S^* \odot s^* S^* u_2^g s^* S \right) = i 1 u_2^g$$

$$\begin{aligned} & \mathbf{u}_2^h \\ & \{s^*, S^*\} \left\{ -\frac{e_2}{2} \right\} \{s^*, S\} \\ & Q = 1 \end{aligned}$$

$$i 1 s^* \odot S^* u_2^h s^* = i \left(1 s^* S^* \odot s^* S^* u_2^h s^* S \right) = i 1 u_2^h$$

$$\begin{aligned} & \mathbf{e}_3 \\ & \{s, S^*\} \{2 e_5 + 2 i e_4\} \{s^*, S^*\} \\ & Q = \frac{2}{3} \end{aligned}$$

$$i \frac{2}{3} s \odot S^* e_3 s = i \left(\frac{2}{3} s S^* \odot s S^* e_3 s^* S^* \right) = i \frac{2}{3} e_3$$

$$\begin{aligned} & \mathbf{v}_3 \\ & \{s^*, S^*\} \{2 i\} \{s, S\} \\ & Q = 1 \end{aligned}$$

$$i 1 s^* \odot S^* v_3 s^* = i \left(1 s^* S^* \odot s^* S^* v_3 s S \right) = i 1 v_3$$

$$\begin{aligned} & \mathbf{d}_3^f \\ & \{s, S^*\} \{0\} \{s^*, S\} \\ & Q = \frac{2}{3} \end{aligned}$$

$$i \frac{2}{3} s \odot S^* d_3^f s = i \left(\frac{2}{3} s S^* \odot s S^* d_3^f s^* S \right) = i \frac{2}{3} d_3^f$$

$$\begin{aligned} & \mathbf{d}_3^g \\ & \{s, S^*\} \{0\} \{s^*, S\} \\ & Q = \frac{2}{3} \end{aligned}$$

$$i \frac{2}{3} s \odot S^* d_3^g s = i \left(\frac{2}{3} s S^* \odot s S^* d_3^g s^* S \right) = i \frac{2}{3} d_3^g$$

$$\begin{aligned} & \mathbf{d}_3^h \\ & \{s, S^*\} \{e_3 + i e_1\} \{s^*, S\} \\ & Q = \frac{2}{3} \end{aligned}$$

$$i \frac{2}{3} s \odot S^* d_3^h s = i \left(\frac{2}{3} s S^* \odot s S^* d_3^h s^* S \right) = i \frac{2}{3} d_3^h$$

$$\begin{aligned} & \mathbf{u}_3^f \\ & \{s, S\} \{0\} \{s^*, S\} \\ & Q = \frac{4}{3} \end{aligned}$$

$$i \frac{4}{3} s \odot S u_3^f s = i \left(\frac{4}{3} s S \odot s S u_3^f s^* S \right) = i \frac{4}{3} u_3^f$$

$$\begin{aligned} & \mathbf{u}_3^g \\ & \{s, S\} \{-e_3 - i e_1\} \{s^*, S\} \\ & Q = \frac{4}{3} \end{aligned}$$

$$i \frac{4}{3} s \odot S u_3^g s = i \left(\frac{4}{3} s S \odot s S u_3^g s^* S \right) = i \frac{4}{3} u_3^g$$

$$\begin{aligned} & \mathbf{u}_3^h \\ & \{s, S\} \{0\} \{s^*, S\} \\ & Q = \frac{4}{3} \end{aligned}$$

$$i \frac{4}{3} s \odot S u_3^h s = i \left(\frac{4}{3} s S \odot s S u_3^h s^* S \right) = i \frac{4}{3} u_3^h$$

Out[2740]=

Checking all SM particle color and flavor changing actions (e.g. $\Lambda_n[v] \odot u1g$) using 4 forms of $\Lambda_{n=1-8}$ and $g_{n=9-14}$ (i.e. with $f \rightarrow v, \bar{v}, v^*, v^\dagger$)

The output is extensive and given my open questions on the formalism presented, the accuracy likely deviates from the intent of [5], but it is interesting to show how everything transforms. If no transform is found for a particular action, it outputs an * for that action. If a color or flavor changing transformation action is found, it identifies that action with the list of particles the transformation applies to. Note: it only identifies a transformed particle if the source particle has a non-zero reduced value and the resulting match is exact (red) or a \pm integer factor of that particle (blue).

```
In[2741]:= Aglist = {"v", "i v", "v*", "v^\dagger"};
```

```
In[2742]:= (* This generates a list of particle reductions
used to compare color and flavor changes against *)
listSM = Flatten[Table[Table[Table[Table[
  prt = ToExpression[flavorList[[flv]] <> ToString@gen <> If[flv > 2, rgb[[col]], " "]];
  (* Conjugate if anti=True *)
  If[anti, prt = prt*];
  sprt = ToString[
    Subsuperscript[flavorList[[flv]], gen, If[flv > 2, rgb[[col]], " "], TraditionalForm];
  If[anti, sprt = Overline[sprt]];
  ssprt = Style[sprt, Bold, Red];
  (* skip color iterations on leptons *)
  If[flv < 3 && col > 1, Nothing,
    {sprt, Expand[prt[[2, 1]] /. f -> fff]}],
  {col, 3}],
  {flv, 4}],
  {gen, 3}],
  {anti, {False, True}}], 3]
```

Out[2742]=

$$\begin{pmatrix}
 e_1 & -i e_1 - 3 e_3 \\
 \nu_1 & 1. + 0. i \\
 d_1^f & -i e_1 - e_3 \\
 d_1^g & 0 \\
 d_1^b & 0 \\
 u_1^f & i e_5 - e_4 \\
 u_1^g & e_3 - i e_1 \\
 u_1^b & -\frac{e_2}{2} \\
 e_2 & 2 i e_2 + 2 e_6 \\
 \nu_2 & 3 e_7 \\
 d_2^f & 2 i e_2 + 2 e_6 \\
 d_2^g & 2 e_7 + 2 i \\
 d_2^b & -2 i e_4 - 2 e_5 \\
 u_2^f & i e_5 - e_4 \\
 u_2^g & e_3 - i e_1 \\
 u_2^b & -\frac{e_2}{2} \\
 e_3 & 2 i e_4 + 2 e_5 \\
 \nu_3 & 2 i \\
 d_3^f & 0 \\
 d_3^g & 0 \\
 d_3^b & i e_1 + e_3 \\
 u_3^f & 0 \\
 u_3^g & -i e_1 - e_3 \\
 u_3^b & 0 \\
 \overline{e_1} & i e_1 - 3 e_3 \\
 \overline{\nu_1} & 1. + 0. i \\
 \overline{d_1^f} & i e_1 - e_3 \\
 \overline{d_1^g} & 0 \\
 \overline{d_1^b} & 0 \\
 \overline{u_1^f} & -e_4 - i e_5 \\
 \overline{u_1^g} & i e_1 + e_3 \\
 \overline{u_1^b} & -\frac{e_2}{2} \\
 \overline{e_2} & 2 e_6 - 2 i e_2 \\
 \overline{\nu_2} & 3 e_7 \\
 \overline{d_2^f} & 2 e_6 - 2 i e_2 \\
 \overline{d_2^g} & 2 e_7 - 2 i \\
 \overline{d_2^b} & 2 i e_4 - 2 e_5 \\
 \overline{u_2^f} & -e_4 - i e_5 \\
 \overline{u_2^g} & i e_1 + e_3 \\
 \overline{u_2^b} & -\frac{e_2}{2} \\
 \overline{e_3} & 2 e_5 - 2 i e_4 \\
 \overline{\nu_3} & -2 i \\
 \overline{d_3^f} & 0 \\
 \overline{d_3^g} & 0 \\
 \overline{d_3^b} & e_3 - i e_1 \\
 \overline{u_3^f} & 0 \\
 \overline{u_3^g} & i e_1 - e_3 \\
 \overline{u_3^b} & 0
 \end{pmatrix}$$

```
In[2743]:= (* This checks one non-zero reduced commutation result (e.g.  $\Delta_n[v] \circ u1g$ ) against all of the non-zero reduced particle values generated in lisSM *)
chkRes@res_ := Module[{res2, pos1p, pos2p, pos3p, pos1m, pos2m, pos3m, pos3pos, pos, pos1, fl},
  (* Generate the position lists for each comparison *)
  res2 = Numerator@# & /@Simplify[ (**) res];
  pos1p = Position[listSM[[All, 2]], Expand[1 res2]];
  pos2p = Position[listSM[[All, 2]], Expand[2 res2]];
  pos3p = Position[listSM[[All, 2]], Expand[3 res2]];
  pos1m = Position[listSM[[All, 2]], Expand[-1 res2]];
  pos2m = Position[listSM[[All, 2]], Expand[-2 res2]];
  pos3m = Position[listSM[[All, 2]], Expand[-3 res2]];
  fl = {pos1p, pos2p, pos3p, pos1m, pos2m, pos3m};
  pos = DeleteDuplicates@Flatten@fl;
  pos1 = DeleteDuplicates@Flatten@pos1p;
  (* If the input is not zero and there was a position found,
  output that particle - color code for red as exact match and blue is a  $\pm$ integer factor *)
  If[res != 0 && Length@pos > 0,
    Style[{listSM[[Flatten@pos, 1]], res},
      If[Length@pos1p > 0 && MemberQ[listSM[[pos1, 2]], res], Red, Blue]],
    Style[{"*", res}, Black]]];
```

```
In[2744]:= (* This generates all of the possible combinations of color and flavor commutations across the SM and lists the checked results *)
doSMActionAll = Module[{prt, sprt, ssprt, pos, res, in},
  Column[Table[Column[Table[Column[Table[Row@Table[
    prt = ToExpression[flavorList[[flv]] <> ToString@gen <> If[flv > 2, rgb[[col]], " "]];
    (* Conjugate if anti=True *)
    If[anti, prt = prt*];
    sprt = ToString[
      Subsuperscript[flavorList[[flv]], gen, If[flv > 2, rgb[[col]], " "], TraditionalForm];
    If[anti, sprt = ToString[sprt, TraditionalForm]];
    ssprt = Style[sprt, Bold, Red];
    (* skip color iterations on leptons *)
    If[flv < 3 && col > 1, Nothing,
      Column[{ssprt,
        Row@{sprt <> "=", Row@{Expand@prt[[2, 1]] /. f -> fff}},
        MatrixForm@Join[
          Table[in = "\Delta[" <> ToString[#] <> ", " <> \Delta glist[[Ag]] <> ""];
            res = doAction[ToExpression@in, prt[[2, 1]]];
            Row@{in, chkRes@res},
            {\Delta g, Length@\Delta glist}] & /@ Range@8,
          Table[in = "g[" <> ToString[#] <> ", " <> \Delta glist[[Ag]] <> ""];
            res = doAction[ToExpression@in, prt[[2, 1]]];
            Row@{in, chkRes@res},
            {\Delta g, Length@\Delta glist}] & /@ Range[9, 14]]], Center]],
    {col, 3}],
  {flv, 4}], Center],
  {gen, 3}], Center],
  {anti, {False, True}}}],
Center]]]
```

$$\begin{array}{cccc}
 \begin{array}{l} A13^7[[4, 4], -1 + i] \\ A12^7[[6, 0] \\ A13^7[[6, -\frac{1}{2} + \frac{i\sqrt{3}}{2}]] \\ A14^7[[6, 0] \\ A15^7[[4, 1], 2 + i] \\ A16^7[[6, 0] \\ A17^7[[6, 0] \\ A18^7[[6, -\frac{1}{2} + \frac{i\sqrt{3}}{2}]] \\ A19^7[[4, 4], \frac{1}{36} + \frac{i\sqrt{3}}{36}] \\ g10^7[[4, 4], -\frac{1}{36} + \frac{i\sqrt{3}}{36}] \\ g11^7[[6, -\frac{1}{12} + \frac{i\sqrt{3}}{12}]] \\ g12^7[[2, 1], 4 - \frac{1}{12} + \frac{i\sqrt{3}}{12}] \\ g13^7[[6, -\frac{1}{12} + \frac{i\sqrt{3}}{12}]] \\ g14^7[[2, 4], \frac{1}{36} + \frac{i\sqrt{3}}{36}] \end{array} &
 \begin{array}{l} A11^7[[1, 4], -1 + i] \\ A12^7[[6, 0] \\ A13^7[[6, -\frac{1}{2} + \frac{i\sqrt{3}}{2}]] \\ A14^7[[6, 0] \\ A15^7[[2, 1], 4 - \frac{1}{12} + \frac{i\sqrt{3}}{12}]] \\ A16^7[[6, 0] \\ A17^7[[6, 0] \\ A18^7[[6, -\frac{1}{2} + \frac{i\sqrt{3}}{2}]] \\ A19^7[[4, 4], \frac{1}{36} + \frac{i\sqrt{3}}{36}] \\ g10^7[[4, 4], -\frac{1}{36} + \frac{i\sqrt{3}}{36}] \\ g11^7[[6, -\frac{1}{12} + \frac{i\sqrt{3}}{12}]] \\ g12^7[[2, 1], 4 - \frac{1}{12} + \frac{i\sqrt{3}}{12}] \\ g13^7[[6, -\frac{1}{12} + \frac{i\sqrt{3}}{12}]] \\ g14^7[[2, 4], \frac{1}{36} + \frac{i\sqrt{3}}{36}] \end{array} &
 \begin{array}{l} A11^7[[2, 2 + i + 2i\sqrt{3}]] \\ A12^7[[6, 0] \\ A13^7[[6, -\frac{1}{2} + \frac{i\sqrt{3}}{2}]] \\ A14^7[[6, 0] \\ A15^7[[2, 1], 4 - \frac{1}{12} + \frac{i\sqrt{3}}{12}]] \\ A16^7[[6, 0] \\ A17^7[[6, 0] \\ A18^7[[6, -\frac{1}{2} + \frac{i\sqrt{3}}{2}]] \\ A19^7[[4, 4], \frac{1}{36} + \frac{i\sqrt{3}}{36}] \\ g10^7[[4, 4], -\frac{1}{36} + \frac{i\sqrt{3}}{36}] \\ g11^7[[6, -\frac{1}{12} + \frac{i\sqrt{3}}{12}]] \\ g12^7[[2, 1], 4 - \frac{1}{12} + \frac{i\sqrt{3}}{12}] \\ g13^7[[6, -\frac{1}{12} + \frac{i\sqrt{3}}{12}]] \\ g14^7[[2, 4], \frac{1}{36} + \frac{i\sqrt{3}}{36}] \end{array} &
 \begin{array}{l} A11^7[[4, 4], -1 + i] \\ A12^7[[6, 0] \\ A13^7[[6, -\frac{1}{2} + \frac{i\sqrt{3}}{2}]] \\ A14^7[[6, 0] \\ A15^7[[4, 1], 2 + i] \\ A16^7[[6, 0] \\ A17^7[[6, 0] \\ A18^7[[6, -\frac{1}{2} + \frac{i\sqrt{3}}{2}]] \\ A19^7[[4, 4], \frac{1}{36} + \frac{i\sqrt{3}}{36}] \\ g10^7[[4, 4], -\frac{1}{36} + \frac{i\sqrt{3}}{36}] \\ g11^7[[6, -\frac{1}{12} + \frac{i\sqrt{3}}{12}]] \\ g12^7[[2, 1], 4 - \frac{1}{12} + \frac{i\sqrt{3}}{12}] \\ g13^7[[6, -\frac{1}{12} + \frac{i\sqrt{3}}{12}]] \\ g14^7[[2, 4], \frac{1}{36} + \frac{i\sqrt{3}}{36}] \end{array}
 \end{array}$$

IX. Addendum : Towards a complete description

Working with Quaternions

In[2745]= $\gamma_1 = \mathbf{I} e_1; \gamma_2 = \mathbf{I} e_2; \gamma_3 = \mathbf{I} e_3; \gamma_4 = \mathbf{I} e_4; \gamma_5 = \mathbf{I} e_5; \gamma_6 = \mathbf{I} e_6; \gamma_7 = \mathbf{I} e_7; \gamma_8 = \mathbf{J} e_7; \gamma_9 = \mathbf{K} e_7;$

In[2746]= **ToQuaternion@ $\gamma_{\#}$ & /@ Range@9**

Out[2746]= {Quaternion(0, e_1 , 0, 0), Quaternion(0, e_2 , 0, 0), Quaternion(0, e_3 , 0, 0), Quaternion(0, e_4 , 0, 0),
Quaternion(0, e_5 , 0, 0), Quaternion(0, e_6 , 0, 0), Quaternion(0, e_7 , 0, 0), Quaternion(0, 0, e_7 , 0), Quaternion(0, 0, 0, e_7)}

In[2747]= **fff@{ $\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6, \gamma_7$ }**

Out[2747]= i

In[2748]= **ToQuaternion[fff@{ $\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6, \gamma_7$ }]**

ToQuaternion@ γ_8

%% ** %

Out[2748]= Quaternion(0, 1, 0, 0)

Out[2749]= Quaternion(0, 0, e_7 , 0)

Out[2750]= Quaternion(0, 0, 0, e_7)

In[2751]= **fff@{ $\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6, \gamma_7$ } ** γ_8**

Out[2751]= $K e_7$

In[2752]= **Table[$\gamma_n \Theta \gamma_m, \{n, 8\}, \{m, 8\}$]**

Out[2752]=
$$\begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & -2iJ \\ 0 & 0 & 0 & 0 & 0 & 0 & -2iJ & -2J^2 \end{pmatrix}$$

The Setup for Multi – Actions

In[2753]= $\mathbf{P}_1 = \frac{1}{2} (\mathbf{1} - \gamma_9)$

Out[2753]= $\frac{1}{2} (1 - K e_7)$

In[2754]= $\mathbf{P}_2 = \frac{1}{2} (\mathbf{1} + \gamma_9)$

Out[2754]= $\frac{1}{2} (K e_7 + 1)$

In[2755]= $\mathbf{m}^\dagger[\mathbf{a}_-, \mathbf{b}_-] := \sum_{i=1}^1 (f@{\mathbf{P}_i, \mathbf{a}} ** f@{\mathbf{P}_i, \mathbf{b}} + f@{\mathbf{P}_i, \mathbf{b}} ** f@{\mathbf{P}_i, \mathbf{a}^\dagger});$

$\tilde{\mathbf{m}}[\mathbf{a}_-, \mathbf{b}_-] := \sum_{i=1}^1 (f@{\mathbf{P}_i, \mathbf{a}} ** f@{\mathbf{P}_i, \mathbf{b}} + f@{\mathbf{P}_i, \mathbf{b}} ** f@{\mathbf{P}_i, \tilde{\mathbf{a}}});$

In[2757]= $\mathbf{m}^\dagger[\mathbf{a}_-, \mathbf{b}_-] := \sum_{i=1}^1 (f@{\mathbf{P}_i, \mathbf{a}, \mathbf{P}_i, \mathbf{b}} + f@{\mathbf{P}_i, \mathbf{b}, \mathbf{P}_i, \mathbf{a}^\dagger});$

$\tilde{\mathbf{m}}[\mathbf{a}_-, \mathbf{b}_-] := \sum_{i=1}^1 (f@{\mathbf{P}_i, \mathbf{a}, \mathbf{P}_i, \mathbf{b}} + f@{\mathbf{P}_i, \mathbf{b}, \mathbf{P}_i, \tilde{\mathbf{a}}});$

In[2759]:= **mt[v, b]**

$$\text{Out[2759]}= f\left(\left\{\frac{1}{2}(1 - Ke_7), b, \frac{1}{2}(1 - Ke_7), -\frac{1}{2} - \frac{ie_7}{2}\right\}\right) + f\left(\left\{\frac{1}{2}(1 - Ke_7), \frac{1}{2}(1 + ie_7), \frac{1}{2}(1 - Ke_7), b\right\}\right)$$

In[2760]:= **mt[v, ω]**

$$\text{Out[2760]}= f\left(\left\{\frac{1}{2}(1 - Ke_7), \frac{1}{2}(-e_7 + i), \frac{1}{2}(1 - Ke_7), -\frac{1}{2} - \frac{ie_7}{2}\right\}\right) + f\left(\left\{\frac{1}{2}(1 - Ke_7), \frac{1}{2}(1 + ie_7), \frac{1}{2}(1 - Ke_7), \frac{1}{2}(-e_7 + i)\right\}\right)$$

In[2761]:= **% /. f -> fff**

$$\begin{aligned} \text{Out[2761]}= & \frac{1}{16} \left((-3 + 2i)K^2e_3 + 3K^2e_4 - (3 - 2i)K^2e_5 - (3 - 2i)K^2e_6 - (2 - i)K^2e_7 + (3 + i)K^2 + (3K + (3 - 2i))Ke_1 + \right. \\ & \left. (3K + (3 - 2i))Ke_2 + 3Ke_3 + (3 - 2i)Ke_4 + 3Ke_5 + 3Ke_6 + (3 + 3i)Ke_7 + (4 - i)K + 2e_7 - 2i \right) + \\ & \frac{1}{16} \left((3 - 2i)K^2e_3 - 3K^2e_4 + (3 - 2i)K^2e_5 + (3 - 2i)K^2e_6 + (2 - i)K^2e_7 - (3 + i)K^2 + (-3K - (3 - 2i))Ke_1 + \right. \\ & \left. (-3K - (3 - 2i))Ke_2 - 3Ke_3 - (3 - 2i)Ke_4 - 3Ke_5 - 3Ke_6 - (3 + 3i)Ke_7 - (4 - i)K - 2e_7 + 2i \right) \end{aligned}$$

In[2762]:= **Simplify@%**

Out[2762]= 0

In[2763]:= **m̃[v, ω]**

$$\text{Out[2763]}= f\left(\left\{\frac{1}{2}(1 - Ke_7), \frac{1}{2}(-e_7 + i), \frac{1}{2}(1 - Ke_7), \frac{1}{2}(1 + ie_7)\right\}\right) + f\left(\left\{\frac{1}{2}(1 - Ke_7), \frac{1}{2}(1 + ie_7), \frac{1}{2}(1 - Ke_7), \frac{1}{2}(-e_7 + i)\right\}\right)$$