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Unimodular rotation of $E_8$ to $H_4$ 600-cells

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We introduce a unimodular Determinant=1 8 × 8 rotation matrix to produce four 4 dimensional copies of $H_4$ 600-cells from the 240 vertices of the Split Real Even $E_8$ Lie group. Unimodularity in the rotation matrix provides for the preservation of the 8 dimensional volume after rotation, which is useful in the application of the matrix in various fields, from theoretical particle physics to 3D visualization algorithm optimization.

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I. INTRODUCTION

Fig. 1 is the Petrie projection of the largest of the exceptional simple Lie algebras, groups and lattices called $E_8$. The Split Real Even (SRE) form has 240 vertices and 6720 edges of 8 dimensional (8D) length $\sqrt{2}$. Interestingly, $E_8$ has been shown to fold to the 4D polychora of $H_4$ (aka. the 120 vertex 720 edge 600-cell) and a scaled copy $H_4\Phi[1][2]$, where $\Phi = \frac{1}{2} (1 + \sqrt{5}) = 1.618...$ is the big golden ratio and $\varphi = \frac{1}{2} (\sqrt{5} - 1) = 1/\Phi = \Phi - 1 = 0.618...$ is the small golden ratio.

FIG. 1: $E_8$ Petrie projection

In my previous papers on the topic [3][4], a specific matrix for performing the rotation of the SRE $E_8$ group of root vertices to the vertices of $H_4$ (a.k.a. the 600-cell) was shown to be that of (1).

$$\begin{pmatrix}
\varphi^2 & 0 & 0 & 0 & \Phi & 0 & 0 & 0 \\
0 & -\varphi & 1 & 0 & 0 & \varphi & 1 & 0 \\
0 & 1 & 0 & -\varphi & 0 & 1 & 0 & \varphi \\
0 & 0 & -\varphi & 1 & 0 & 0 & \varphi & 1 \\
\Phi & 0 & 0 & 0 & \varphi^2 & 0 & 0 & 0 \\
0 & \varphi & 1 & 0 & 0 & -\varphi & 1 & 0 \\
0 & 1 & 0 & \varphi & 0 & 1 & 0 & -\varphi \\
0 & 0 & \varphi & 1 & 0 & 0 & -\varphi & 1 \\
\end{pmatrix}$$ (1)

The convex hull of two opposite edges of a regular icosahedron forms a golden rectangle (as shown in Fig. 2). The twelve vertices of the icosahedron can be decomposed in this way into three mutually-perpendicular golden rectangles, whose boundaries are linked in the pattern of the Borromean rings. Columns 2-4 of $H_4\text{fold}$ contains 6 of the 12 vertices of this icosahedron, including 2 at the origin (with the other 6 of 12 icosahedron vertices being the reflection of these through the origin).

FIG. 2: The Icosahedron formed from 3 mutually-perpendicular golden rectangles

The trace of this matrix is $2(\varphi^2 - \varphi + 1) = 1.527$ and its determinant $Det = (2\sqrt{\varphi})^8 = 37.349$. 

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Notice that $H_{4\text{fold}} = H_{4\text{fold}}^T$ such that it is symmetric with a quaternion-octonion Cayley-Dickson-like structure.

Only the first 4 rows are needed for folding $E_8$ to $H_4$ by dot product with each vertex. This results in two copies of $H_4$ scaled by $\Phi$. Using the full matrix to rotate $E_8$ results in not two, but four copies of $H_4$ 600-cell associated with the two scaled copies ($H_4$ and $H_4\Phi$) and the right (R) 4 dimensions associated with another two copies ($H_4$ and $H_4\Phi$). Rotation back to $E_8$ is achieved with a rotation matrix of $H_4^{-1}$.

II. THE UNIMODULARITY FACTOR

The Platonic solid icosahedral symmetry establishes some valuable utility in this particular construction of $H_{4\text{fold}}$. Yet, the non-unimodularity of the determinant causes the resulting 8D volume of the objects involved in a rotation (or projection) between $E_8 \leftrightarrow H_4$ to vary. In order to correct this, while keeping the general structure of the matrix the same, we simply divide the matrix by a factor of $2\sqrt{\phi}$, giving a $\text{Det} = 1$. This gives:

$$H_{4\text{uni}} = \begin{pmatrix} \sqrt{\phi} & 0 & 0 & 0 & \frac{1}{\sqrt{\phi}} & 0 & 0 & 0 \\ 0 & -\sqrt{\phi} & \frac{1}{\sqrt{\phi}} & 0 & 0 & \sqrt{\phi} & \frac{1}{\sqrt{\phi}} & 0 \\ 0 & \frac{1}{\sqrt{\phi}} & 0 & -\sqrt{\phi} & 0 & \frac{1}{\sqrt{\phi}} & 0 & \sqrt{\phi} \\ 0 & 0 & \sqrt{\phi} & 0 & 0 & 0 & -\sqrt{\phi} & \frac{1}{\sqrt{\phi}} \\ \frac{1}{\sqrt{\phi}} & 0 & 0 & 0 & \sqrt{\phi} & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{\phi}} & 0 & \sqrt{\phi} & 0 & \frac{1}{\sqrt{\phi}} & 0 & -\sqrt{\phi} \\ 0 & 0 & \sqrt{\phi} & \frac{1}{\sqrt{\phi}} & 0 & 0 & \sqrt{\phi} & \frac{1}{\sqrt{\phi}} \end{pmatrix} / 2$$

III. $H_{4\text{fold}}$ FROM 2 QUBIT QUANTUM COMPUTING CNOT AND SWAP GATES

Looking at the four quadrants of $H_{4\text{fold}}$ and $H_{4\text{uni}}$, we see that they resemble a combination of the unitary Hermitian matrices commonly used for Quantum Computing (QC) qubit logic, namely those of the 2 qubit CNOT (3) and SWAP (4) gates. Taking these patterns, combined with the recursive functions that build $\Phi$ from the Fibonacci sequence, it is straightforward to derive both $H_{4\text{fold}}$ and $H_{4\text{uni}}$ from scaled QC logic gates.

$$\text{CNOT} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad (3)$$

$$\text{SWAP} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (4)$$

The code to establish CNOT and SWAP implementations of $H_{4\text{fold}}$ is naively done (in Mathematica™ code) as shown in Fig. 3.

```mathematica
SWAP + @ CNOT & /@ {\(\phi, \phi\)};
Flatten /@ Transpose@Join[
  {Flatten[\%, 1]},
  {Flatten[Reverse@\%, 1]}]
```

FIG. 3: Producing $H_{4\text{fold}}$ from 2 Qubit CNOT and SWAP QC Gates

More interestingly, we can produce a similar result using a recursive function for $\Phi$ using the Fibonacci sequence. This is shown in Figs. 5-7 in Appendix A, where we iterate the Mathematica™ Fibonacci function $n = 10$ times. As $n \to \infty$, the matrix resolves to $H_{4\text{fold}}$ or $H_{4\text{uni}}$.

The numerical result for the first 4 rows of $H_{4\text{fold}}$ is shown in Fig. 4 at $n = 10$.

```mathematica
rndMat = mat
```

FIG. 4: Numerical result for the first 4 rows of $H_{4\text{fold}}$ from the 2 Qubit CNOT and SWAP QC gates and an integer Fibonacci series function output after $n = 10$ iterations

IV. CONCLUSION

Instead of simply folding the 8D $E_8$ vertices into 4D pairs of $H_4$ and $H_4\Phi$ vertices, we rotate them using an $8 \times 8$ matrix. This transforms $E_8$ into a fourfold $H_4$ 600-cell structure. We show that bringing unimodularity to the folding matrix with a $\text{Det} = 1$ is a simple modification. We also show that the folding matrix can easily be generated using 2 qubit QC matrices and recursive functions related to the Fibonacci sequence.
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Appendix A: Mathematica™ code

Producing the first 4 rows of $H_{4\text{fold}}$ from 2 Qubit CNOT and SWAP QC gates and an integer Fibonacci series function

```
fb = Fibonacci;
im = IdentityMatrix;
nC@0 := CNOT.SWAP;
nC@1 := nC[0]';
nC@i_ := nC[i-2] + nC[i-1];
nCInv := Inverse@nC[#]' &;
{mat = Join[
    fb[# + 1] / fb[#] (2 nC[# - 1].nCInv@# - im@4)'[[{1, 4, 3, 2}]],
    fb[# - 1] / fb[# + 1] (2 nC[# + 1].nCInv@# + nC@1)'[[{1, 3, 2, 4}]]'
] & /@ Range@10}
```
FIG. 6: Integer Fibonacci series function output for each of 
$n = 10$ iterations
FIG. 7: Convex hulls of the recursively derived Icosahedron points with volumes

\[
\begin{align*}
&1.33333, & 23.1667, & 1.62997, & 5.62047, & 3.63679, \\
&4.31743, & 4.04676, & 4.14856, & 4.10944, & 4.12435
\end{align*}
\]